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THE ANDERSON–BROWN–PETERSON SPLITTING OF SPIN BORDISM (& REALITY)

BACKGROUND

- $\mathrm{Spin}(n) \xrightarrow{2:1} SO(n)$ nontrivial double cover
- $\gamma_{\mathrm{Spin}} \longrightarrow B\mathrm{Spin}(n)$ universal bundle
- Thom space $M\mathrm{Spin}(n)$
- Ring spectrum $M\mathrm{Spin}$
- Pontjagin–Thom construction:

$$\begin{array}{ccc} \pi_* M\mathrm{Spin} & \cong & \Omega_*^{\mathrm{Spin}} \\ \| & & \| \\ M\mathrm{Spin}_* & & \left\{\begin{array}{l} \text{bordism ring of spin} \\ \text{manifolds under } \amalg \text{ \& } \times \\ \text{graded by } * = \text{dimension} \end{array}\right\} \end{array}$$

- A manifold M has a spin structure iff $w_2(M) = 0$ & $\{\text{spin structures}\} \cong H^1(M; \mathbb{Z}/2)$.
- Spin manifolds play a significant role in the study of global differential geometry (e.g. scalar curvature & smooth structures).

[**GOAL**: Describe Ω_*^{Spin} via simpler invariants.

[**STRATEGY**: Study $M\mathrm{Spin}$.

IN CONTEXT

$$\begin{tikzcd}
 & MString \arrow[d] & & \\
MSU \arrow[r] \arrow[d] & MSpin \arrow[d] & & \\
MU \arrow[r] & MSpin^c \arrow[r] & MSO \arrow[r] & MO
\end{tikzcd}$$

SPIN BORDISM & TOP'L K-THEORY

[The Atiyah-Bott-Shapiro construction]

Def: The Clifford algebra of $\mathbb{R}^n$ is the initial unital $\mathbb{Z}/2$-graded algebra w/ a linear map $\mathbb{R}^n \longrightarrow Cl_n^{odd}$ s.t. $v^2 = -\langle v, v\rangle$.

$\bullet\!\!\rightarrow$ $Spin(n) \cong \langle v \in \mathbb{R}^n \subseteq Cl_n \mid \|v\| = 1 \rangle^{even} \subseteq Cl_n$

The construction

$W = W^0 \oplus W^1$ $\mathbb{Z}/2$-graded Cl_n-module

$$\begin{tikzcd}
\mathbb{R}^n \times W^0 \arrow[r, "\varphi_W"', "(v,x)\mapsto(v,v\cdot x)"] \arrow[d] & \mathbb{R}^n \times W^1 \arrow[d] \\
\mathbb{R}^n \arrow[r, "="'] & \mathbb{R}^n
\end{tikzcd}$$

$\Rightarrow$ $\varphi_W : \underline{W}^0 \longrightarrow \underline{W}^1$ s.t. $\varphi_W|_{\mathbb{R}^n - \{0\}}$ is an isom.

$\rightarrow$ $[\varphi_W] \in KO(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \pi_n(KO)$

Thm (ABS): This construction is additive & multiplicative & surjects onto $\pi_* KO$. (ABS only verify for $* \geq 0$, but true $\forall * \in \mathbb{Z}$.)

i.e. Clifford modules represent KO-classes & $\oplus \mapsto +$ & $\otimes \mapsto \cdot$

Given $P \to B$ a principal $\mathrm{Spin}(n)$-bundle & Δ an irred. Cl_n-module, can form

$$\$ = P \times_{\mathrm{Spin}(n)} \Delta$$

called the <u>spinor bundle</u>.

This construction results in a map of ring spectra:

$$MSpin \xrightarrow{\Phi^{\Gamma}} KO$$

which has been used extensively to:

(1) Obtain Thom isomorphisms in KO for spin bundles.

$\boxed{\text{(2) Approximate spin bordism via KO-theory.}}$

<u>Remark</u>: There is a classical bordism invariant

$$\begin{tikzcd}
\Omega^{SO}_* \arrow[r, "\hat{A}"] & \mathbb{Q} \\
\Omega^{spin}_* \arrow[u] \arrow[r, "\Phi^{\Gamma}_*"] \arrow[ur, phantom, "\circlearrowleft"] & KO_* \arrow[u]
\end{tikzcd}$$

defined via a power series.

$\Rightarrow \hat{A}([M]_{spin}) \in \mathbb{Z}$

& $\hat{A}([M]_{spin}) = 0$ if $\dim(M) \neq 4n$.

<u>The complex case</u>

$$\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \otimes U(1) \subset Cl_n \otimes \mathbb{C}$$

$\Longrightarrow$ ring spectrum map

$$MSpin^c \xrightarrow{\Phi^c} KU$$

Similarly, $\Phi^c_* \cong Td : \Omega^U_* \longrightarrow \mathbb{Z}$.

KO – CHARACTERISTIC CLASSES

[à la Anderson–Brown–Peterson.]

- $U(1)^m \subseteq SO(2m)$ maximal torus
- $X_j : BU(1)^m \xrightarrow{\text{Proj}_j} BU(1)$ line bundle
- $x_j = [X_j] - 1 \in KU^0(BU(1)^m)$
- $\prod_{j=1}^{m} (1 + t(x_j + \bar{x}_j)) \in KU^0(BU(1)^m)[t]$

$$\parallel$$

$$\sum_i \tilde{\pi}^i t^i, \quad \text{where} \quad \tilde{\pi}^i \in KU^0(BU(1)^m).$$

<u>Prop</u>: The element $\tilde{\pi}^i$ is in the image of the monomorphism

$$KO^0(BSO(2m)) \longrightarrow KU^0(BSO(2m)) \longrightarrow KU^0(BU(1)^m)$$

<u>Def</u>: The <u>ith KO-Pontrjagin class</u>, $\pi^i \in KO^0(BSO(2m))$ is the preimage of $\tilde{\pi}^i$.

- π^i is stable in $m \Rightarrow \pi^i \in KO^0(BSO)$

  i.e. $\pi^i : BSO \longrightarrow KO$.

<u>Thm</u> (ABP): Let $ph_i : KO^0(X) \to H^{4i}(X; \mathbb{Q})$ be the ith component of the Pontrjagin (Chern) character, & let $P_i \in H^{4i}(BSO; \mathbb{Q})$ be the ith Pontrjagin class. Then,

- $ph_i(\pi^i) = P_i$ &
- $ph_i(\pi^j) = 0$ for $i < j$.

Some applications of KO-characteristic classes

Thm (ABP): $(\Omega_*^{SU})_{torsion}$ is an $\mathbb{F}_2$-vector space with a specified basis.

Thm (ABP): The image of the unit

$$\begin{array}{ccc} \mathbb{S}_* & \xrightarrow{u_*} & MSU_* \\ \rotatebox{90}{$\cong$} & & \rotatebox{90}{$\cong$} \\ \Omega_*^{framed} & \longrightarrow & \Omega_*^{SU} \end{array}$$

is
$$\mathrm{Im}(u_*) = \begin{cases} \mathbb{Z}/2, & k = 1, 2 \mod 8 \\ 0, & \text{else} \end{cases}.$$

Thm (ABP): The Kervaire invariant of an $(8k+2)$-dimensional framed manifold is zero.

$\boxed{\text{KO-C.C.'s ON SPIN BUNDLES}}$

[AND THE 2-LOCAL SPLITTING OF MSpin]

- $BSpin \longrightarrow BSO \xrightarrow{\pi^i} KO$
- Given a partition $I = (i_1, \dots, i_k)$, let
  $$\pi^I = \pi^{i_1} \cdots \pi^{i_k} : BSpin \longrightarrow KO.$$
- $\forall n \in \mathbb{Z}$ $\exists$ functor $\tau_n : Sp \longrightarrow Sp$
  s.t. $$\pi_i(\tau_n X) = \begin{cases} 0, & i < n \\ \pi_i X, & i \geq n \end{cases}.$$
  with a natural map $$X\langle n \rangle := \tau_n X \longrightarrow X.$$

Thm (ABP): Let $|I| = \sum_{l=1}^{k} i_l$ & let

$$n_I = \begin{cases} 4|I|, & |I| \text{ even} \\ 4|I|-2, & |I| \text{ odd}. \end{cases}$$

Then the map π^I lifts to $KO\langle n_I \rangle$:

$$\begin{tikzcd} & KO\langle n_I\rangle \arrow[d] \\ BSpin \arrow[ur, dashed, "\exists"] \arrow[r, "\pi^I"'] & KO \end{tikzcd}.$$

We can lift the maps π^I further to the Thom spectrum $MSpin$ via the ABS orientation.

Def: Given a vector bundle $V \to B$, the Thom diagonal is the map

$$Th(V) \xrightarrow{\Delta} Th(V) \wedge B$$

induced by the vector bundle map

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V \times 0 \arrow[d] \\ B \arrow[r] & B \times B \end{tikzcd}$$ on Thom spaces.

Def: Let $f_I : MSpin \to KO\langle n_I \rangle$ be the composite

$$MSpin \xrightarrow{\Delta} MSpin \wedge BSpin \xrightarrow{\Phi^c \wedge \pi^I} KO\langle 0 \rangle \wedge KO\langle n_I \rangle \xrightarrow{\mu} KO\langle n_I \rangle$$

Idea: Put all the f_I's together to practically determine $MSpin$.

Thm (ABP): There exist elements $Z \subseteq H^*(MSpin; \mathbb{Z}/2)$ such that the map

$$MSpin \xrightarrow{(\bigvee_I f_I) \vee (\bigvee_z z)} \Big(\bigvee_{\substack{I \in \mathcal{P} \\ \text{s.t. } i_k \neq 1}} KO\langle n_I \rangle \Big) \vee \Big(\bigvee_{z \in Z} \Sigma^{|z|} H\mathbb{Z}/2 \Big)$$

is a 2-local equivalence.

idea: Study the mod 2 cohomology of BSpin & $KO\langle n_I \rangle$ as modules over the Steenrod algebra.

Since $MSpin_*$ has no $p \neq 2$ torsion, this splitting allows for a complete computation of $MSpin_*$.

Cor: $[M] = [N] \in \Omega_*^{Spin}$ iff all KO-Pontryagin numbers & Stiefel-Whitney numbers of M & N agree.

Cor: Let $u_*: \Omega_*^{Spin} \to \Omega_*^{SO}$. Then

$$\begin{cases} \ker(u_*) \neq 0, & * = 1,2 \mod 8 \ \text{(described explicitly)} \\ \ker(u_*) = 0, & * \neq 1,2 \mod 8 \end{cases}$$

Cor: Let $u_*: \Omega_*^{framed} \longrightarrow \Omega_*^{Spin}$. Then

$$\mathrm{Im}(u_*) = \begin{cases} \mathbb{Z}/2, & k = 1,2 \mod 8 \\ 0, & \text{else}. \end{cases}$$

Cor (Stong): $\exists$ ring monomorphism: $\Omega_*^{Spin}/\text{torsion} \longrightarrow \mathbb{Z}[b_1, b_2, \ldots]$ w/ $|b_i| = 4i$.

$\boxed{\text{Reality}}$ [Real spin bordism]

Stong proved a spin^c version of the ABP splitting: $\exists\ Z \subseteq H^*(MSpin^c;\mathbb{Z}/2)$ s.t.

$$MSpin^c \longrightarrow \bigvee_{I\in P} KU\langle 4|I|\rangle \vee \bigvee_{z\in Z} \Sigma^{|z|} H\mathbb{Z}/2$$

is a 2-local equivalence.

$C_2 \circlearrowright U(1) \quad \rightsquigarrow \quad C_2 \circlearrowright Spin^c(n) = Spin(n) \times_{\{\pm 1\}} U(1)$

w/ $Spin(n) = Spin^c(n)^{C_2}$.

Thm (Halladay-K): $\exists\ C_2$-ring spectrum $MSpin^c_{\mathbb{R}}$ with ring maps

- $MSpin^c = (MSpin^c_{\mathbb{R}})^e$
- $MSpin \longrightarrow (MSpin^c_{\mathbb{R}})^{C_2}$
- $\underbrace{MSpin^c_{\mathbb{R}} \longrightarrow KU_{\mathbb{R}}}$.

This map allows us to refine the ABP construction C_2-equivariantly.

From here on, all C_2-spectra are taken in Sp^{BC_2}.

e.g. let $KU_{\mathbb{R}}\langle n\rangle$ denote the composite

$$BC_2 \xrightarrow{KU_{\mathbb{R}}} Sp \xrightarrow{\tau_n} Sp .$$

Thm (Abdallah-K): $\exists$ map in Sp^{BC_2},

$$MSpin^c_{\mathbb{R}} \longrightarrow \bigvee_{I} KU_{\mathbb{R}}\langle 4|I|\rangle \vee \bigvee_{z} \Sigma^{|z|} H\mathbb{Z}/2$$

w/ trivial $C_2 \circlearrowright H\mathbb{Z}/2$ whose underlying map is ABP/Stong's.

Cor: There is a 2-local equivalence

$$\left(MSpin^c_{\mathbb{R}}\right)^{hC_2} \longrightarrow \bigvee_I KU_{\mathbb{R}}\langle 4|I|\rangle^{hC_2} \vee \bigvee_z \left(\Sigma^{|z|} H\mathbb{Z}/2\right)^{hC_2}$$