Spin bordism, Reality, and C^*-algebras

joint with:
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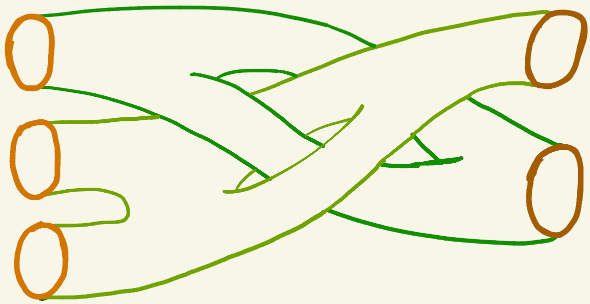
Bordism rings of manifolds

$$\Omega_n^{O} = \left\{ \begin{matrix} n\text{-dimensional} \\ \text{closed manifolds} \end{matrix} \right\} \Big/ \text{bordism}$$

$S^1 \amalg S^1 \amalg S^1$



$S^1 \amalg S^1$

$\hookrightarrow$ abelian group under $\amalg$

$\Omega_* = \bigoplus_n \Omega_n$ (graded) commutative ring under $\times$.

<u>Idea</u>: Study (structures on) manifolds via more computable invariants

$$\text{Manifolds} \longrightarrow \Omega_*^{G} \xrightarrow[\text{“genus”}]{} E_*$$

with structure group G (e.g. SO, Spin, U)

“genus” = ring homomorphism

Example: $\hat{A} : \Omega^{SO}_* \longrightarrow \mathbb{Q}$.

$(\mathrm{Spin}(n) \xrightarrow{2:1} SO(n))$

$$\begin{tikzcd}
\Omega^{SO}_* \arrow[r, "\hat{A}"] & \mathbb{Q} \\
 & \mathbb{Z} \arrow[u, phantom, "\subseteq"] \\
\Omega^{Spin}_* \arrow[uu] \arrow[ur, "\text{(Atiyah-Singer)}"] \arrow[r, "\alpha", "\text{(Atiyah-Bott-Shapiro)}"'] & KO_* \arrow[u]
\end{tikzcd}$$

$$KO_* = \begin{cases} \mathbb{Z}, & * = 0 + 8k \\ 2\mathbb{Z}, & * = 4 + 8k \\ \mathbb{Z}/2, & * = 1, 2 + 8k \\ 0, & \text{else} \end{cases}$$

Index of Dirac operator (Atiyah-Singer index thm) $= \alpha$

Thm: (Lichnerowicz '63, Hitchin '74, Gromov-Lawson '80, Stolz '92)
A simply connected spin manifold M, $\dim(M) \geq 5$, admits a metric with positive scalar curvature

$$\iff \alpha(M) = 0$$

Example: $\alpha(S^9) = 0$ but $\exists$ spin manifold $\Sigma^9 \underset{\text{homeo}}{\cong} S^9$ with $\alpha(\Sigma^9) \neq 0 \in \mathbb{Z}/2$

To prove such a theorem, need more structure than just a genus $\Omega^G_* \longrightarrow E_*$.

Spectra (stable homotopy theory)

$$\mathrm{Sp}$$

"Spaces/dimension" $\simeq$ Sp $\simeq$ "Generalized cohomology/homology theories"

"Spaces/dimension" $\approx$ $(\text{Spaces}, \times)[\{\mathbb{R}^n\}^{-1}]$

"Generalized cohomology/homology theories" $\approx$ linear homotopy invariants of spaces.

X space $\Rightarrow$ suspension spectrum

$$\Sigma^\infty_+ X \approx \begin{pmatrix} \vdots \\ \mathrm{Map}(S^1, X) \\ X \\ S^1 \wedge X \\ S^2 \wedge X \\ \vdots \end{pmatrix}$$

E^* cohomology thy

$$E^*(X) \cong [X, E(n)]$$

$$E \approx \begin{pmatrix} \vdots \\ E(-1) \\ E(0) \\ E(1) \\ \vdots \end{pmatrix}$$

Spectra have (abelian) homotopy groups:

$$E_* = \pi_* E = \pi_{*+k} E(k) \qquad (k \gg 0)$$

Thom spectra

$$G(n) \longrightarrow O(n) \qquad \text{"compatible"}$$

$\Rightarrow$ universal rank n vector bundle

$$\gamma_{G(n)} = \begin{pmatrix} EG(n) \times_{G(n)} \mathbb{R}^n \\ \downarrow \\ BG(n) \end{pmatrix}$$

Thom space $MG(n) = \mathrm{Thom}(\gamma_{G(n)})$

$\Rightarrow$ (commutative ring) spectrum $MG = \{MG(n)\}_{n \geq 0}$.

Pontryagin-Thom Theorem/Philosophy:

$$MG_* \cong \Omega^G_*$$

$$MG_*(X) = \{M \hookrightarrow X\}/\sim$$

$$MG^*(X) \approx \left\{ \begin{matrix} M \to \mathcal{F} \\ \downarrow \\ X \end{matrix} \right\}/\sim$$

Def: A <u>G orientation</u> of a ring spectrum R is a ring map
$$MG \longrightarrow R$$

↳ Taking homotopy groups gives a genus i.e. bordism invariant.

↳ Provides Thom isomorphisms
$$R^*(X) \xrightarrow[\cdot u]{\sim} R^{*+\mathrm{rank}V}(V)$$
for vector bundles $V \to X$ with G-structure.

<u>Examples:</u> ① $\exists$ orientation $MSO \longrightarrow H\mathbb{Z} \longrightarrow H\mathbb{R}$.

↳ $M^{\text{oriented}} \hookrightarrow X \xrightarrow{f} \mathbb{R} \Rightarrow \int_M f \in \mathbb{R}$

↳ "fundamental class" $MSO_*(X) \longrightarrow H_*(X;\mathbb{Z})$

↳ oriented vector bundles have Thom classes.

② $MO \longrightarrow H\mathbb{Z}/2$.

↳ all manifolds have a fundamental class mod 2.

<u>Topological K-theory</u>

$A = \mathbb{R}, \mathbb{C}$ (more generally, a C^*-algebra)

$\exists$ spectrum $K(A) = KO, KU$ with

$K(A)^0(X) =$ "stable" A-vector bundles on X.

Atiyah-Bott-Shapiro:

$$\begin{tikzcd}
 & MSpin \arrow[r] \arrow[d] & KO \arrow[d, "\otimes \mathbb{C}"] \\
MU \arrow[r] \arrow[urr, dashed, red, "\nexists"] & MSpin^c \arrow[r] \arrow[d] & KU \arrow[d, "ch"] \\
 & MSO \arrow[r] \arrow[d] \arrow[ur, dashed, red, "\nexists"] & H\mathbb{Z} \arrow[d] \\
 & MO \arrow[r] \arrow[ur, dashed, red, "\nexists"] & H\mathbb{Z}/2
\end{tikzcd}$$

where

- $Spin^c(n) =$ "$Spin(n) \otimes \mathbb{C}$" $\cong \underset{\{\pm 1\}}{Spin(n) \times U(1)}$
- $$\begin{tikzcd} U(n) \arrow[r, hook] & Spin^c(2n) \arrow[d] \\ & SO(2n) \arrow[u, phantom, "\subseteq"] \end{tikzcd}$$

Determining bordism spectra via orientations

Thom ('54): $MO \simeq \prod_{z} \Sigma^{|z|} H\mathbb{Z}/2$ ← "shift of homotopy groups"

Anderson-Brown-Peterson ('67):

$$MSpin \xrightarrow[\text{2-local}]{\sim} \prod_{I} KO\langle n_I \rangle \times \prod_{z} \Sigma^{|z|} H\mathbb{Z}/2$$

ABP, Stong ('68):

$$MSpin^c \xrightarrow[\text{2-local}]{\sim} \prod_{I} KU\langle 4|I| \rangle \times \prod_{z} \Sigma^{|z|} H\mathbb{Z}/2$$

Buchanan-Mckean ('23/'25):

$$MSpin^h \xrightarrow[\text{2-local}]{\sim} \prod_{I} fib_I KSp\langle 4|I| \rangle \times \prod_{z} \Sigma^{|z|} H\mathbb{Z}/2$$

$\lvert I \rvert$	fib_I
even	fib_*
odd	$fib_{\mathbb{Z}/2}$

Reality: $\mathbb{C}_{\mathbb{R}} = \left(\mathbb{C} \circlearrowleft \overline{(\ )} \in C_2\right)$. $\quad \mathbb{C}_{\mathbb{R}}^{C_2} = \mathbb{R}$.

$\Rightarrow$ **Real vector bundle:**

$$\begin{tikzcd} \mathbb{C}^n_{\text{“}\mathbb{R}\text{”}} \arrow[r] & V \circlearrowleft C_2 \ni \underset{e}{\overset{c}{\neq}} \text{ acts conjugate-linearly.} \arrow[d] \\ & X \circlearrowleft C_2 \end{tikzcd}$$

Atiyah: $\quad KU_{\mathbb{R}} = K(\mathbb{C}_{\mathbb{R}}) \qquad C_2\text{-spectrum}$

underlying: $\quad KU_{\mathbb{R}}^{e} = KU$

fixed points: $\quad KU_{\mathbb{R}}^{C_2} = KO \simeq KU_{\mathbb{R}}^{hC_2}$

Similarly, $\quad U(n) \circlearrowleft \overline{(\ )} \rightsquigarrow MU_{\mathbb{R}}$ &

C_2-orientation $\quad MU_{\mathbb{R}} \to KU_{\mathbb{R}}$.

⚠
- $MU_{\mathbb{R}}^{C_2} \simeq MU_{\mathbb{R}}^{hC_2} \not\simeq MO \simeq MU_{\mathbb{R}}^{gC_2}$
- $\Omega_*^{Real} \xrightarrow{\not\cong} \pi_* MU_{\mathbb{R}}$ but it's close (transversality issues)

Theorem: (Halladay-K) $\exists$ C_2-commutative ring spectrum $MSpin^c_{\mathbb{R}}$, Real spin bordism, with

$$\begin{tikzcd} & MSpin \arrow[r] \arrow[d, "\not\cong"'] & KO \arrow[d, equal] \\ & (MSpin^c_{\mathbb{R}})^{C_2} \arrow[r] \arrow[d] & KU_{\mathbb{R}}^{C_2} \arrow[d] \\ MU_{\mathbb{R}} \arrow[r] & MSpin^c_{\mathbb{R}} \arrow[r] & KU_{\mathbb{R}} \end{tikzcd}$$

<u>Idea</u>: C_2-representation $V \Rightarrow$

$$\begin{array}{ccc} C_2 & & \overline{(\ )} \\ \circlearrowright & & \circlearrowright \\ \mathrm{Spin}(V) & \times_{\{\pm 1\}} & U(1) \end{array} =: \mathrm{Spin}^c_{\mathbb{R}}(V)$$

$$\mathrm{Spin}(n) \xrightarrow{\sim} \mathrm{Spin}^c_{\mathbb{R}}(n)^{C_2}$$

$$U_{\mathbb{R}}(n) \xrightarrow[C_2]{} \mathrm{Spin}^c_{\mathbb{R}}(n\rho) = \mathrm{Spin}^c_{\mathbb{R}}(\mathbb{C}^n_{\mathbb{R}})$$

$$V = \mathbb{R}^n \Rightarrow C_2 \subset \Omega^{\mathrm{Spin}^c}_*$$

conjugates the det. line bundle of a spin^c-manifold

i.e. negates the first chern class

<u>Thm*</u>: (Abdallah-K) The ABP-Stong splitting of $M\mathrm{Spin}^c$ refines to a map of spectra with C_2-action

$$M\mathrm{Spin}^c_{\mathbb{R}} \longrightarrow \prod_I KU_{\mathbb{R}}\langle 4|I|\rangle \times \prod_z \Sigma^{|z|} H\mathbb{Z}/2$$

<u>Cor</u>: (homotopy fixed points)

$$(M\mathrm{Spin}^c_{\mathbb{R}})^{hC_2}_* \cong \bigoplus_I ko_*[\delta_I] \oplus \bigoplus_z H^{-*+|z|}(\mathbb{RP}^\infty; \mathbb{Z}/2)$$

⚠ The components $M\mathrm{Spin}^c_{\mathbb{R}} \to KU_{\mathbb{R}}\langle 4|I|\rangle$ for $|I|$ odd are not maps of genuine C_2-spectra; however, we suspect they refine to genuine maps

$$M\mathrm{Spin}^c_{\mathbb{R}} \longrightarrow KU_{\mathbb{R}}\langle 4|I|, 2\rangle.$$

* one calculation is still being worked out.

C^*-algebras & naturality of spin orientations

(real super) C^*-algebra $A \longmapsto K(A) \in \mathrm{Mod}_{KO}\mathrm{Sp}$

"K is the initial homotopy invariant, semi-exact, functor on $C^*\mathrm{Alg}$ stable under $-\otimes(\text{compact operators})$".

$$\mathcal{U}(A) = \{a \in A \mid a^*a = aa^* = 1\}$$

$$\mathrm{Spin}^A(n) = \mathrm{Spin}(n) \times_{\{\pm 1\}} \mathcal{U}(A)^{ev} \amalg \mathrm{Pin}(n)^{odd} \times_{\{\pm 1\}} \mathcal{U}(A)^{odd}$$

$\Rightarrow \quad M\mathrm{Spin}^A \in \mathrm{Mod}_{M\mathrm{Spin}}(\mathrm{Sp})$. (Let $\dim A < \infty$ now)

Theorem (Moores–K): $\exists$

$$\begin{tikzcd}
sC^*\mathrm{Alg}^{f.d.} \arrow[rr, bend left, "M\mathrm{Spin}^{(\ )}"] \arrow[dr, "K"'] & \Downarrow\alpha & \mathrm{Mod}_{M\mathrm{Spin}}(\mathrm{Sp}) \\
& \mathrm{Mod}_{KO}(\mathrm{Sp}) \arrow[ur] &
\end{tikzcd}$$

pullback along ring map $M\mathrm{Spin} \to KO$

Such that $\alpha_A : M\mathrm{Spin}^A \to K(A)$ is the spin, spin^c, & spin^h orientations when $A = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

e.g. this shows $M\mathrm{Spin}^h \to KSp$ is a strict $M\mathrm{Spin}$-module map.

Other examples

- $MSpin^{Cl_{\pm 1}} \simeq MPin^{\mp} \longrightarrow \Sigma^{\pm 1} KO$
- $MSpin^{Cl_{-2}} \simeq MPin^{c}$
- $MSpin^{M_n(\mathbb{C})} =$ "$MSpin\text{-}U(n)$"

Positive scalar curvature

M spin simply connected $\dim \geq 5$

$$\left(\exists \text{ p.s.c.m.}\right) \iff \left(\begin{array}{ccc} MSpin_* & \longrightarrow & KO_* \\ [M] & \longmapsto & 0 \end{array}\right)$$

M spin w/ $\pi = \pi_1(M)$, $\dim \geq 5$

$$\begin{tikzcd} \tilde{M} \arrow[d, "\tilde{f}"'] & & & & \\ M \arrow[r, "f"] & B\pi & & & \end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd} {[M,f] \in MSpin_*(B\pi)} \arrow[r, dotted, "?"', "\text{``Assembly map?''}"] \arrow[d, "ABS"'] \arrow[dr, "\alpha_\pi"] & MSpin^{C^*_\pi}_* \arrow[d, "\alpha_{C^*_\pi}"] \\ KO_*(B\pi) \arrow[r, "\text{Assembly map}"'] & K(C^*_\pi)_* \end{tikzcd}$$

(Baum-Connes Conj)

Gromov-Lawson-Rosenberg Conjecture:

$$\left(\exists \text{ p.s.c.m}\right) \iff \left(\alpha_\pi([M,f]) = 0\right)$$

↳ known for various finite groups and free & free abelian groups. known to be false for some infinite groups. Still open for general finite groups.