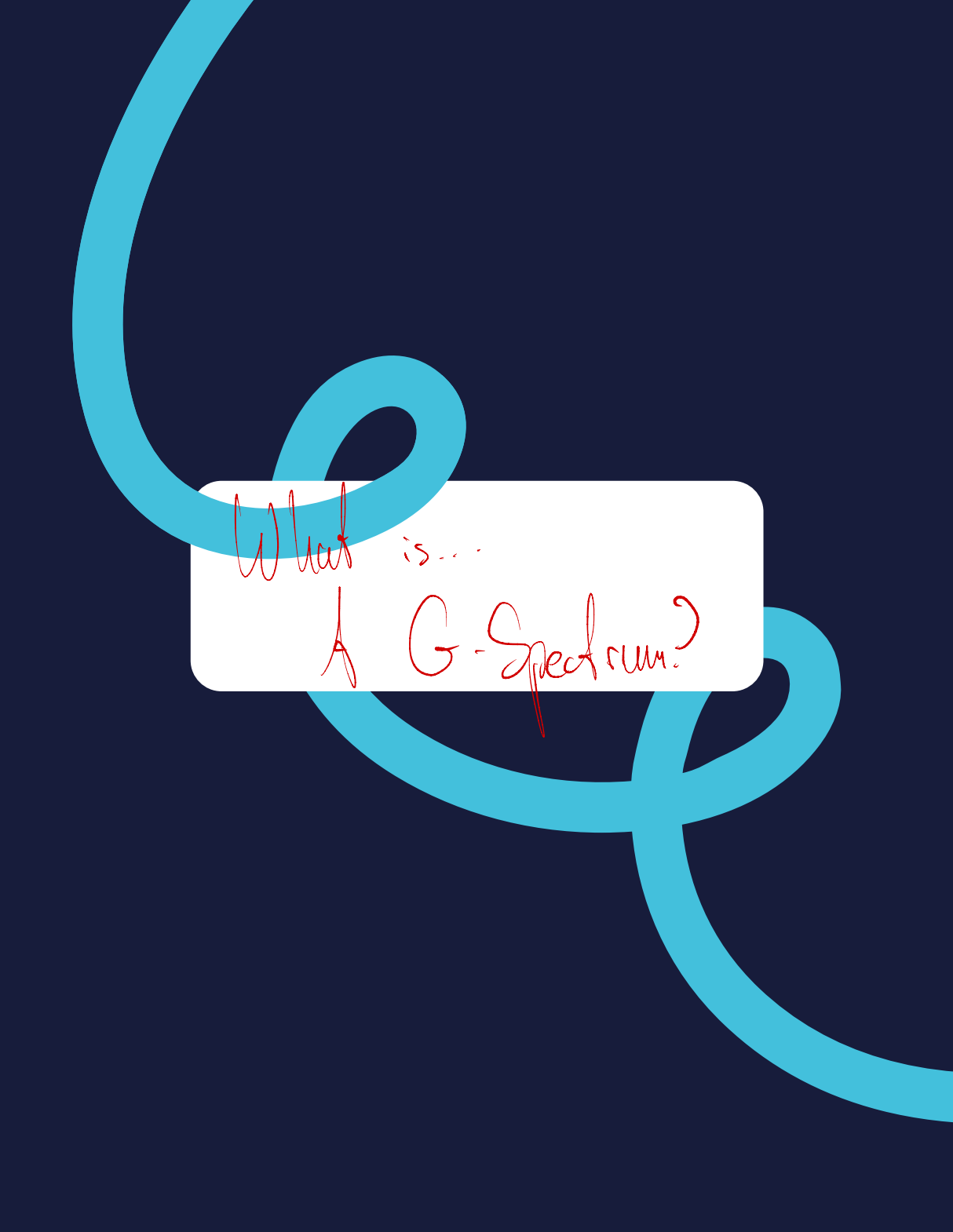




What is...

A G-Spectrum?

2-13-24) G-Spectra

This talk: Formulate stable equivariant category.

Talk structure:

1. What should G-spectra even be?

2.1 Sequential / G-prespectra

- No genuine monoidal structure
- can upgrade notion to diagram spectra

3). Symmetric Spectra

- sym monoidal
- Day convolution
- W.E. not Π isos

4) Orthogonal Spectra

- $hO(G)$ graded brown rep
- Even better symmetries.

Sequential Spectra / Prespectra.

Defn A prespectrum is a sequence of spaces X_n w/ structure maps $f_n: \Sigma X_n \rightarrow X_{n+1}$.

Maps of prespectra are levelwise maps commuting w/ structure maps.

How to "equivariantify"? Lets just give this G -action later!

Defn G -prespectrum: Seqn of spaces X_n w/ G -actions s.t. structure maps are G -equivariant.

Just like how $T_G \neq T^G$, can choose maps to be levelwise equivariant or not.

Also called **Naive G -spec / Spec w/ G -action**

Talk later, we defined G -spectra!

However, this notion of spectra does not nearly have all of the properties we want & even has full on problems.

What do we want out of our spectra anyway?

For one, our category should be closed symmetric monoidal.

And naive G-spectra are *very* non symmetric monoidal.

Given $\{X_n\}, \{Y_n\}$ spectra we want to define

$X \wedge Y \sim \{X_i \wedge Y_j\}_n$ which could be

$$X_n \wedge Y_n \sim \{X_i \wedge Y_j\}_n \quad (i+j=n)$$

$$\bigvee_{i+j=n} X_i \wedge Y_j, \quad \begin{cases} X_i \wedge Y_i, & n=2i \\ S^1 X_i \wedge Y_i, & n=2i+1 \end{cases}$$

1 & 2 aren't "symmetric"

3 requires many choices & Adams spent 30 pages showing it only makes $h\mathbb{Z}/2\mathbb{Z}$ into a sym monoidal cat

4, 5) hence no good canonical structure maps.

In a sense, because $\Omega^\infty \Sigma^\infty X \cong \Omega X$, there can be no good smash product on sequential spectra.

How to improve? Well one can reinterpret the data of
as a diagram functor over the sphere spectrum.
 $f_n: \Sigma x_n \rightarrow x_{n+1}$
following MMS

Explicitly:

$\mathcal{U}$: Category w/ objs $[n]$ indexed by integers w/
 $|\text{Hom}([n], [m])| = \delta_n^m$.

It is symmetric monoidal w/ $[n] \otimes [m] = [n+m]$.

it is an example of a

Topological Category: Monoidal category $\mathcal{C}$ monoidally
enriched over $\mathcal{T}op$

$\mathcal{C}$ -spectrum) for $\mathcal{C}$ topologically enriched is a functor
 $X: \mathcal{C} \rightarrow \mathcal{T}op$

So an $\mathcal{U}$ -spectrum gives us the set of x_n , what gives the structure
maps?

Assume we have some C -spectrum S "generalizing" the sphere spectrum $\mathbb{S}$ s.t. it is monoidal, i.e. we have a natural trans. $\varphi: S(-) \wedge S(-) \Rightarrow S(- \otimes -)$ and a map $\lambda: S^0 \rightarrow S(1_C)$

C -spectrum over S)

is a C -spectrum X w/ a natural transformation $\sigma: X(-) \wedge S(-) \Rightarrow X(- \otimes -)$

s.t.

$X(d) \xrightarrow{\sim} X(d) \wedge S^0 \xrightarrow{id \wedge \lambda} X(d) \wedge S(1_C) \xrightarrow{\sim} X(d \otimes 1_C) \xrightarrow{\sim} X(d)$
is ident and

$$\begin{array}{ccc} X(d) \wedge S(e) \wedge S(f) & \xrightarrow{\sigma \sim id} & X(d \otimes e) \wedge S(f) \\ \downarrow id \wedge \lambda & \searrow \sigma & \downarrow \sigma \\ X(d) \wedge S(e \otimes f) & \xrightarrow{\sigma} & X(d \otimes e \otimes f) \end{array}$$

Now $S: \mathcal{M} \rightarrow \mathcal{T}^G$
is a monoidal functor ∇ for X a $\mathcal{M}$ -spectrum over S
the maps σ are exactly

$$\sigma: X(m) \wedge S(n) = X_n \wedge S^n = \Sigma^n X_m \rightarrow X(m+n) = X_{m+n}$$

Thus $\mathcal{M}^G$, the category of $\mathcal{M}$ -spectra over S_{an} is exactly the category of $\mathcal{U}$ -spectrum!

Why this language?

Thm 1 Day Convolution

Given $(\mathcal{C}, \otimes_{\mathcal{C}})$ monoidal cat enriched over $(\mathcal{V}, \otimes)$ closed sym monoidal. get canonical monoidal structure on $\text{Fun}(\mathcal{C}, \mathcal{V})$, and it is symmetric monoidal if $\otimes_{\mathcal{C}}$ is.

$$F, G \in \text{Fun}(\mathcal{C}, \mathcal{V}) \text{ then } (F \otimes G) = \int^{x, y \in \mathcal{C}} \text{Hom}_{\mathcal{C}}(x \otimes y, -) \otimes Fx \otimes Gy$$

or a left Kan Extension

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{(F \otimes G) \circ (-)} & \mathcal{T}^G \\ & \searrow \otimes & \uparrow \\ & \mathcal{C} & F \otimes G \end{array}$$

Thus for $\mathcal{C}$ a topological category, this gives a (sym) monoidal product on $\text{Fun}(\mathcal{C}, \mathcal{T}^G)$

What about for $\mathcal{C}$ -spectrum over S_{an} ?

What is the right action $\sigma: (x \wedge y)(m) \wedge S(m) \rightarrow (x \wedge y)(m)$

$$\begin{array}{ccc}
 x \wedge y \wedge s & \xrightarrow{id \wedge \sigma} & x \wedge y \\
 \searrow id \wedge \tau & & \nearrow \sigma \wedge id \\
 & x \wedge s \wedge y &
 \end{array}$$

we have two possible maps, can make $x \wedge y$ the coequalizer of the two, e.g. like in modules.

Note that to give maps $(x \wedge s, y) \wedge s \wedge s \rightarrow (x \wedge s, y) \wedge s$

$$\begin{array}{ccc}
 (x \wedge s, y) \wedge s \wedge s & \rightarrow & (x \wedge s, y) \wedge s \\
 \downarrow & & \downarrow \\
 (x \wedge s, y) \wedge s & \rightarrow & (x \wedge s, y)
 \end{array}$$

to commute, the twisting involved in $x \wedge s, y$ "forces" S to be commutative, i.e.

$$\begin{array}{ccc}
 S(c, c) \wedge S(c, c) & \xrightarrow{\varphi} & S(c, c \otimes c) \\
 \downarrow \quad \quad \downarrow S(c, \tau) & & \\
 S(c, c) \wedge S(c, c) & \xrightarrow{\varphi} & S(c, c \otimes c)
 \end{array}$$

Answer: This explains why presheaves don't have a good smash product!

$\tau: (m+n) = (m) \otimes (n) \rightarrow (m) \otimes (n) = (m+n)$ is the identity while $S^n \wedge S^m \rightarrow S^m \wedge S^n$ is not the identity!

This suggests that we should allow nonidentity morphisms in our indexing category, i.e. more symmetries!

Symmetric Category One could repeat this w/

Source category Σ : objs $[n] = \{1, \dots, n\}$

$\text{Hom}([n], [n]) = \Sigma_n$, empty otherwise

Our functor $S_\Sigma: \Sigma \rightarrow T^G$ takes $\{1, \dots, n\}$ to $\prod_{i=1}^n s' \approx s^n$
takes a permutation $\sigma \in \Sigma_n$ to the map that permutes the factors.

This almost by defn makes S_Σ into a commutative monoid, so
Symmetric G -action get symmetric smash product!

Problem

- 1) Stable equivalences will not preserve homotopy groups /
not strict weak equiv $\neq$ Π_* isos
- 2) G -action not embedded into category.

Orthogonal spectra

"should represent $HO(G)$ -graded coh. theories"

just like how Brown rep suggested def. lets examine what should happen.

went a next time

$$\begin{array}{ccc} E^*(-) & \rightsquigarrow & E^{*+\nu}(\Sigma^\nu -) \\ \text{"} & & \text{"} \\ [-, E] & & [\Sigma^\nu -, E^{*+\nu}] \end{array}$$

yoneda gives $\int E \rightsquigarrow \Omega^\nu E^{*+\nu}$ or equiv

$$\int_\nu: \Sigma^\nu E \rightarrow E^{*+\nu}$$

this works for any $i: V \hookrightarrow U \oplus U$, but these maps are a bit uninteresting to consider categorically.

instead, to $i: U \hookrightarrow W$ we want to associate

$$\int_{U,W}: \Sigma^? E^U \rightarrow E^W.$$

? needs to be some complement W/U . If they are orthogonal reps & i is isom. embed, $W - i(U)$ orthogonal complement is free!

Orthogonal Cat: $\mathcal{J}$, obj's are vec spaces,
 $\text{Hom}_{\mathcal{J}}(U, V) = \mathcal{O}(U, V)$,
 sym prop is $U \otimes V$.

Given orthogonal spectra.

(hmk) Since vector spaces are classified by d.m, sym. spec
 can be seen as orthogonal spec w/ choice of basis ($U \approx \mathbb{R}^n$).
 No wonder they are a bit ugly!

However, we still haven't baked in AdG -goodness!

recall, to every iso metric embedding $i: U \hookrightarrow W$, we want to
 associate a map $\mathcal{J}: S^{W-U} \wedge E^U \rightarrow E^W$.

This is adjoint to $S^{W-U} \rightarrow \text{Hom}(E^U, E^W)$

$$\text{eg. } S^{W-U} \rightarrow E[\text{Hom}_{\mathcal{J}}(U, 0)]$$

Thus we could also require $\text{Hom}_{\mathcal{J}}(U, W)$ to "give S^{W-U} for any $i: U \hookrightarrow W$ "
 However that is exactly what $\text{Thom}(\mathcal{O}(U, W); W-U)$ is!

$\mathcal{O}(U, W) =$ Steif manifold of linear isometric embeddings $i: U \hookrightarrow W$

$W - U$ is a vector bundle over this space, where $(W - U)_i = W - i(U)$ is the orthogonal complement.

and taking the Thom space makes it a sphere bundle! Thus.

Mandell May Category: $\mathcal{J}_G$

• Objects, orthogonal fin dim G -reps U (in some universe)

• $\text{Hom}_{\mathcal{J}_G}(U, W) = \text{Thom}(\mathcal{O}(U, W); W - U)$

This carries a G -action, which is fiberwise the G -action on S^{W-U}

This is symmetric monoidal, and $\downarrow$ shows it's enriched over $\mathcal{T}^G$.

Thus Day Convolution gives sym. smooth product on $\text{Fun}(\mathcal{J}_G, \mathcal{T}^G)$ enriched functors.

G -action encoded in this enrichment. $\mathcal{J}_G$ enriched over $\mathcal{T}^G$
 $\mathcal{T}^G$ enriched over $\mathcal{T}^G$

so functor should resp this prop.

Thus we get orthogonal G -spectra!

map $O(V) \times E^V \rightarrow E^V$ given by

$$E(O \cup V, V) = E(O \cup V)$$

and compat. cond. encoded in sym monoidal struct