PHD DEFENCE TALK: K-theory, spin bordism, coherence, and Reality.

Ordinary (co)homology

$\{\amalg \Delta^n \to X\} \longrightarrow H_n(X)$

$\Omega^n(X) \longrightarrow H^n(X)$ ★

↳ "oriented manifold shapes in X":

$MSO_n(X) \longrightarrow H_n(X)$

$\Rightarrow \quad MSO \longrightarrow H\mathbb{Z}$

$\& \quad MO \longrightarrow H\mathbb{Z}/2$

Atiyah-Bott-Shapiro ('60s)

★ (Clifford modules) $\longrightarrow$ (Top'l K-theory classes)

1

ring isomorphisms

$\pi_* KO \cong M_*/M_{*+1}$

$\pi_* KU \cong M^c_*/M^c_{*+1}$

+ coherence

2

ring spectra maps

$MSpin \longrightarrow KO$

$MSpin^c \longrightarrow KU$

+ Reality

The ABS Construction

★ I will write using real notation, but everything works for complex K-theory too

⚠ For expository purposes: I will NOT be consistent on Clifford sign conventions

$P \to X$ = $Spin(n)$-bundle

W = Cl_n-module

$\Rightarrow \quad \$ = P \times_{Spin(n)} W \longrightarrow X$

$\Rightarrow \quad \underbrace{\{Cl_n\text{-modules}\}^{gp}}_{= \mathcal{M}_n} \longrightarrow KO^0(V, V-0)$

$V = P \times_{Spin(n)} \mathbb{R}^n$

Thm: (ABS) $X = *$:

$$\bigoplus_{n\geq 0} \mathcal{M}_n \xrightarrow{\varphi} \bigoplus_{n\geq 0} KO^0(S^n) \cong \bigoplus_{n\geq 0} \pi_n KO$$

is surjective & $\varphi(W) = 0$ iff the Cl_n-module structure on W extends to $Cl_{n+1} \cong Cl_n \otimes Cl_1$

Further,

$$\otimes : \mathcal{M}_p \times \mathcal{M}_q \longrightarrow \mathcal{M}_{p+q}$$

& $\displaystyle\bigoplus_n \mathcal{M}_n / \mathcal{M}_{n+1} \cong \bigoplus_n \pi_n KO$ of rings.

For any X, $n = 8k$:

$\Rightarrow$ Thom class $u_V \in KO^0(V, V-0) \cong KO^{8k}(Th(V))$.

$\boxed{\text{1. Coherence}}$ Refine to a full description of KO as a spectrum / cohomology theory.

Def: (Hohnhold-Stolz-Teichner, Cheung) The <u>n^{th} topological Atiyah-Bott-Shapiro category</u>, $\mathrm{Mod}^{nd}_{top}(Cl_n)$, has

<u>objects</u>: f.d. Cl_n-modules (w/ inner product)

<u>morphisms</u>: $W_0 \xhookrightarrow{\imath} W_1$ isometric embedding with an extension of Cl_n-module structure on $W_1 \ominus \imath W_0$ to $Cl_n \otimes Cl_{-1}$.

Thm: (HST, C) The classifying space $B\mathrm{Mod}^{nd}_{top}(Cl_n)$ represents $KO^n(-)$.

↳ Not yet a spectrum / cohomology theory.

<u>Hope</u>: Functors?

$$\mathrm{Mod}^{nd}_{top}(Cl_p) \times \mathrm{Mod}^{nd}_{top}(Cl_q) \xrightarrow{\otimes} \mathrm{Mod}^{nd}_{top}(Cl_{p+q})$$

<u>No</u>: Reasonable candidates for such a functor are not functorial.

↳ similar to Thomason's "Beware the phony multiplication on Quillen's $S^{-1}S$"

Solution: N^{hc}: top'l cats $\longrightarrow$ quasicategories.

$$\mathrm{Mod}^{nd}_{\infty}(\mathcal{C}\ell_n) := N^{hc}\mathrm{Mod}^{nd}_{top}(\mathcal{C}\ell_n)$$

Thm: (K.) $\exists$ semi-simplicial maps

$$\mathrm{Mod}^{nd}_{\infty}(\mathcal{C}\ell_p) \times \mathrm{Mod}^{nd}_{\infty}(\mathcal{C}\ell_q) \xrightarrow{m_{p,q}} \mathrm{Mod}^{nd}_{\infty}(\mathcal{C}\ell_{p+q})$$

given by $\otimes$ on objects.

Thm: (K.) $\exists$ commutative symmetric ring spectrum, ABS, w/ spaces given by $|\mathrm{Mod}^{nd}_{\infty}(\mathcal{C}\ell_n)|$ & multiplication given by $|m_{p,q}|$, such that

$$ABS \simeq KO.$$

Cor: For any spin vector bundle V, the ABS construction produces a Thom class

$$u_V \in KO^{\mathrm{rank}(V)}(Th(V)).$$

$\boxed{2. \text{ Reality}}$

"$\mathbb{R} + \mathbb{C}$" $\approx$ "$\mathbb{C}_{\mathbb{R}} \circlearrowleft \overline{(\,)}$"

$$MSpin \longrightarrow KO \simeq KU_{\mathbb{R}}^{C_2}$$

$$MU \longrightarrow MSpin^c \longrightarrow KU \simeq KU_{\mathbb{R}}^{e}$$

$$MU_{\mathbb{R}} \longrightarrow ? \longrightarrow KU_{\mathbb{R}} = KR$$

Thm: (Halladay–K.) $\exists$ $MSpin^c_{\mathbb{R}} \in CAlg(Sp^{C_2})$, called <u>Real spin bordism</u>, with underlying spectrum $MSpin^c$, & a diagram in $CAlg(Sp^{C_2})$

$$\begin{tikzcd} & MSpin \arrow[r] \arrow[d, "\nexists\,?"'] & KO \arrow[d, phantom, "\cong" rotate=90] \\ & (MSpin^c_{\mathbb{R}})^{C_2} \arrow[r] \arrow[d] & KU_{\mathbb{R}}^{C_2} \arrow[d] \\ MU_{\mathbb{R}} \arrow[r] & MSpin^c_{\mathbb{R}} \arrow[r] & KU_{\mathbb{R}} \end{tikzcd}$$

Thm: (Abdallah–Halladay–K.) $\nexists$ $E \in Alg(Sp^{C_2})$ w/ $E^e \simeq MSpin^c \in Alg(Sp)$ & $E^{C_2} \simeq MSpin \in Sp$.

★ ASK FOR SHORT PROOF!

$$\Rightarrow \quad (MSpin^c_{\mathbb{R}})^{hC_2} \not\simeq MSpin.$$

The homotopy fixed points

What do $MSpin$ & $MSpin^c$ even look like?

<u>Anderson–Brown–Peterson</u>

$$\begin{pmatrix} \text{ABS} \\ \text{orientations} \end{pmatrix} + \begin{pmatrix} \text{K-theory} \\ \text{characteristic} \\ \text{classes} \end{pmatrix} \Rightarrow \begin{pmatrix} \text{2-local splitting} \\ \text{of} \\ MSpin \text{ \& } MSpin^c \end{pmatrix}.$$

$i \in \mathbb{Z}_{\geq 1}$ $\quad \exists \quad$
$$\begin{tikzcd} \pi^i_r \arrow[d, mapsto] \in KO^0(BSO) \arrow[d, "\otimes \mathbb{C}"] \\ \pi^i \in KU^0(BSO) \end{tikzcd}$$

$$I = (i_1, \dots, i_k) \implies \pi^I_{(r)} := \pi^{i_1}_{(r)} \cdots \pi^{i_k}_{(r)}$$

Thm: (ABP, Stong) $\quad n_I = \begin{cases} 4|I|, & |I| \text{ even} \\ 4|I|-2, & |I| \text{ odd} \end{cases}$

& $\quad |I| = \sum_{l=1}^{k} i_l \implies \exists$ lifts:

$$\begin{tikzcd} & ko\langle n_I \rangle \arrow[d] \\ BSO \arrow[ur, dashed, "\exists"] \arrow[r, "\pi^I_r"'] & KO \end{tikzcd} \quad \& \quad \begin{tikzcd} & ku\langle 4|I| \rangle \arrow[d] \\ BSO \arrow[ur, dashed, "\exists"] \arrow[r, "\pi^I"'] & KU \end{tikzcd}$$

Will just carry Spin^c case on, but similar for Spin.

$$\begin{tikzcd} M\mathrm{Spin}^c \arrow[r, "\Delta"] \arrow[drr, "f^I"'] & M\mathrm{Spin}^c \wedge B\mathrm{Spin}^c \arrow[r, "ABS \wedge \pi^I"] & ku \wedge ku\langle 4|I| \rangle \arrow[d, "\mu"] \\ & & ku\langle 4|I| \rangle \end{tikzcd}$$

Thm: (ABP, S) $\exists\ Z \subseteq H^*(M\mathrm{Spin}^c; \mathbb{Z}/2)$ s.t.

$$M\mathrm{Spin}^c \xrightarrow{(\vee f^I) \vee (\vee z)} \bigvee_{I \in P} ku\langle 4|I| \rangle \vee \bigvee_{z \in Z} \Sigma^{|z|} H\mathbb{Z}/2$$

is a 2-local equivalence.

All torsion in $\pi_* M\mathrm{Spin}^c$ is 2-torsion

$\implies$ computes $M\mathrm{Spin}^c_*$.

Thm: (Abdallah–K.) $\exists$ map in Sp^{BC_2}

$$(MSpin^c_{\mathbb{R}})^e \longrightarrow \bigvee_I ku_{\mathbb{R}}\langle 4|I|\rangle^e \vee \bigvee_z \Sigma^{|z|} H\mathbb{Z}/2$$

whose underlying map is ABP-S's above.

Cor: $\exists$ 2-local equivalence,

$$(MSpin^c_{\mathbb{R}})^{hC_2} \longrightarrow \bigvee_I ku_{\mathbb{R}}\langle 4|I|\rangle^{hC_2} \vee \bigvee_z (\Sigma^{|z|} H\mathbb{Z}/2)^{hC_2}$$

$\Rightarrow$ explicit description of $\pi_*(MSpin^c_{\mathbb{R}})^{hC_2}$

Genuine refinement? $\rightsquigarrow (MSpin^c_{\mathbb{R}})^{C_2}$

How we got Borel equivariance:

$$\begin{tikzcd}
BSpin^c_{\mathbb{R}} \arrow[r, "\pi^I"] \arrow[d] & ku_{\mathbb{R}}\langle 4|I|\rangle^e \\
BSO \arrow[r] & ko\langle n_I\rangle \arrow[u]
\end{tikzcd}$$

- $|I|$ even: $n_I = 4|I|$. ✓ ABP.
- $|I|$ odd: $n_I = 4|I|-2$:

$$\begin{tikzcd}
 & ko\langle 4|I|\rangle \arrow[r] \arrow[d] & ku_{\mathbb{R}}\langle 4|I|\rangle^e \arrow[d] \\
BSO \arrow[r, "ABP"'] \arrow[ur, dashed, "\text{ABP: } \nexists"] & ko\langle 4|I|-2\rangle \arrow[r] \arrow[ur, "\exists c"] & ku_{\mathbb{R}}\langle 4|I|-2\rangle^e
\end{tikzcd}$$

Proposition: $\exists \ ku_{\mathbb{R}}\langle 4n, 2\rangle \in Sp^{C_2}$

with:

underlying $\qquad\qquad$ C_2- fixed points

$$\begin{tikzcd} \overline{(\ )} \circlearrowright ku\langle 4n\rangle \arrow[rr, shift right] \arrow[dr] & & ko\langle 4n-2\rangle \arrow[ll, shift right, "c"'] \\ & ko\langle 4n\rangle \arrow[ur] & \end{tikzcd}$$

★ ASK FOR SHORT PROOF!

Cor: The characteristic class $\pi^{\mathcal{I}} \in KU^0(BSO)$ refines to a genuine

$$\begin{cases} BSO \longrightarrow ku_{\mathbb{R}}\langle 4|I|\rangle, & |I| \text{ even} \\ BSO \longrightarrow ku_{\mathbb{R}}\langle 4|I|, 2\rangle, & |I| \text{ odd}. \end{cases}$$

Follow-up work

(1) Does the ABP map refine to Sp^{C_2}?

$\quad\hookrightarrow$ if so, how far from equivalence?

(2) "Real" coherent ABS construction.

(3) Use $\longrightarrow\!\!\!\!\!\!\!\!\!\!\!\!\;\;\;\;\;\mathcal{I}$ to model differential $KU_{\mathbb{R}}$.

(4) Real spin^{A} orientations

THANKS!