REAL SPIN BORDISM & ORIENTATIONS OF TOPOLOGICAL K-THEORY

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joint work with

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ORIENTATIONS OF RING SPECTRA

MAPS OF RING SPECTRA $MG \to E$	$\leftrightsquigarrow$	THOM ISOMORPHISMS IN E-COHOMOLOGY FOR "G"-VECTOR BUNDLES

ORDINARY COHOMOLOGY: $MO \longrightarrow H\mathbb{Z}/2$

$MSO \longrightarrow H\mathbb{Z}$

"fundamental class"

TOPOLOGICAL K-THEORY:

$$MU \longrightarrow MSpin^c \longrightarrow KU = (KU_{\mathbb{R}})^{\{e\}}$$

$$MSpin \longrightarrow KO = (KU_{\mathbb{R}})^{C_2}$$

$$MU_{\mathbb{R}} \longrightarrow \boxed{?} \longrightarrow KU_{\mathbb{R}} = KR$$

EVIDENCE FOR $\boxed{?}$ VIA THOM ISOMORPHISMS

$\mathbb{R}^{p,q}$ = C_2-representation

$\overset{\text{universal property}}{\Longrightarrow}$ $C_2 \circlearrowright Cl(\mathbb{R}^{p,q})$ $\overset{\text{complex conjugation}}{\Longrightarrow}$ $C_2 \circlearrowright \mathbb{C}l(\mathbb{R}^{p,q})$

$$\cup\!\!\mid$$

restricts to $C_2 \circlearrowright Spin^c(p,q)$

Theorem (Atiyah): $G \longrightarrow Spin^c(p,q)$ C_2-equivariant

$\Rightarrow KU_{\mathbb{R},G}(X) \xrightarrow[\cdot u]{\sim} KU_{\mathbb{R},G}(\mathbb{R}^{p,q} \times X)$, $p \equiv q \mod 8$.

↳ contains all Thom isomorphisms above,

e.g. $Spin(8n) \hookrightarrow Spin^c(8n,0)$

& $U(n) \hookrightarrow Spin^c(n,n)$

C_2-equivariantly

Ex:

$$\begin{array}{ccc} KO_{Spin(8n)}(ESpin(8n)) & \cong & KO_{Spin(8n)}(ESpin(8n) \times \mathbb{R}^{8n}) \\ \cong & & \cong \\ KO(BSpin(8n)) & \cong & KO(\Upsilon_{Spin(8n)}) \end{array}$$

THEOREM: (Halladay–K) $\exists$ C_2-E_∞-ring spectrum $MSpin^c_{\mathbb{R}}$, Real spin bordism, such that

(1) $MSpin^c = (MSpin^c_{\mathbb{R}})^{\{e\}}$

(2) $\exists$ E_∞-map $MSpin \longrightarrow (MSpin^c_{\mathbb{R}})^{C_2}$

(3) $\exists$ C_2-E_∞-maps $MU_{\mathbb{R}} \xrightarrow{(a)} MSpin^c_{\mathbb{R}} \xrightarrow{(b)} KU_{\mathbb{R}}$

$\boxed{\text{Real spin orientation}}$

$(1)+(3b) \implies MSpin^c \longrightarrow KU$

$(2)+(3b) \implies MSpin \longrightarrow KO$

$(3a)+(3b) \implies MU_{\mathbb{R}} \longrightarrow KU_{\mathbb{R}}$

Picture

$$
\begin{tikzcd}
 & MSpin \arrow[r] \arrow[d] & KO \arrow[d, "\wr"] \\
 & (MSpin^c_{\mathbb{R}})^{C_2} \arrow[r] \arrow[d] & (KU_{\mathbb{R}})^{C_2} \arrow[d] \\
MU_{\mathbb{R}} \arrow[r] & MSpin^c_{\mathbb{R}} \arrow[r] & KU_{\mathbb{R}}
\end{tikzcd}
$$

THE FIXED POINTS OF $MSpin^c_{\mathbb{R}}$

<u>Recall:</u>

$$\begin{tikzcd} Spin(n) \arrow[rr] \arrow[dr, "\cong"'] & & Spin^c(n) \\ & Spin^c(n)^{C_2} \arrow[ur, hook] & \end{tikzcd}$$

(2)

$$\begin{tikzcd} MSpin \arrow[rr, "u"] \arrow[dr, "\exists w"'] & & MSpin^c \\ & (MSpin^c_{\mathbb{R}})^{C_2} \arrow[ur, "res"'] & \end{tikzcd}$$

<u>Theorem (G):</u> (Halladay-K)

Any w factoring u is not an equivalence.

Proof of (G) uses

$$\begin{tikzcd} MSO_* \arrow[r, "\hat{A}"] & \mathbb{Q} \\ MSpin_* \arrow[u] \arrow[r, "\hat{A}"] & \mathbb{Z} \arrow[u, hook] \end{tikzcd}$$

$\Rightarrow$ only applies to ω s.t. $MSpin \xrightarrow{\text{res} \circ \omega} MSpin^c$ preserves the $\hat{A}$-genus (which the natural map does.)

Theorem (A): (Abdallah-Halladay-K) $\nexists$ C_2-ring spectrum M, s.t. $M^{\{e\}} \simeq MSpin^c$ & $M^{C_2} \simeq MSpin$.

$\simeq$ as ring spectra

$\simeq$ as spectra

Cor: $(MSpin^c_{\mathbb{R}})^{C_2} \not\simeq MSpin$

proof (A): π_*-Mackey functor

Ring map

$$\mathrm{MSpin}_* \underset{tr}{\overset{res}{\rightleftarrows}} \mathrm{MSpin}^c_* \circlearrowleft C_2 \ni \overline{(\ )}$$

Facts: I. $\forall x \in \mathrm{MSpin}^c_*$, $x + \bar{x} = res(tr(x))$

II. $\exists$ subring $\mathbb{Z}[\beta, \gamma] \subseteq \mathrm{MSpin}^c_*$, $|\beta| = 2$, $|\gamma| = 4$.

III.

$*$	2	4	6
MSpin_*	$\mathbb{Z}/2$	$\mathbb{Z}$	0

$\Rightarrow$

$x = \beta$ in I $\Rightarrow \bar{\beta} = -\beta$

$x = \gamma \Rightarrow \bar{\gamma} = n\beta^2 - \gamma$

$x = \beta\gamma \Rightarrow \beta\gamma + \overline{\beta\gamma} = -n\beta^3 + 2\beta\gamma \neq 0$ ※

*under certain conditions.

Proof comments

- Joachim: $MSpin^c_{\mathbb{R}} \to KU_{\mathbb{R}}$ (added) "$\Rightarrow$" (1), (2), (3b)

- (3a) uses Lemma*: $(X \times_{U(n)} Y)^{C_2} \cong X^{C_2} \times_{O(n)} Y^{C_2}$

  ↳ nontrivial: pf uses that symmetric unitary matrices have symmetric unitary square roots.

Remark: $Spin^c(n) = Spin(n) \times_{\{\pm 1\}} U(1) \circlearrowleft$ complex conjugation

$\Rightarrow$ spectrum with C_2-action $(MSpin^c_{\mathbb{R}})^h$

$\rightsquigarrow$ underlying spectrum w/ C_2-action ("cofree completion") $\Rightarrow MSpin^c_{\mathbb{R}} \xrightarrow{\overset{?}{\sim}} (MSpin^c_{\mathbb{R}})^h$

$\Rightarrow$ all results apply to $(MSpin^c_{\mathbb{R}})^h$ as well.

QUESTIONS & FUTURE WORK (ongoing with Hassan Abdallah)

- Is the map $MSpin^c_{\mathbb{R}} \to (MSpin^c_{\mathbb{R}})^h$ an equivalence?

  i.e. is $(MSpin^c_{\mathbb{R}})^{C_2} \longrightarrow (MSpin^c_{\mathbb{R}})^{hC_2}$ an equivalence?

- [Anderson-Brown-Peterson] 2-local equivalences

$$MSpin \simeq \bigvee_{i \in I} ko\langle i \rangle \vee \bigvee_{y \in Y} \Sigma^{|y|} H\mathbb{Z}/2$$

$$MSpin^c \simeq \bigvee_{j \in J} ku\langle j \rangle \vee \bigvee_{z \in Z} \Sigma^{|z|} H\mathbb{Z}/2$$

Refine to $MSpin^c_{\mathbb{R}}$?

THANKS FOR LISTENING!

$$
\begin{tikzcd}
 & MSpin \arrow[r] \arrow[d, "\not\wr"] & KO \arrow[d, "\wr"] \\
 & (MSpin^c_{\mathbb{R}})^{C_2} \arrow[r] \arrow[d] & (KU_{\mathbb{R}})^{C_2} \arrow[d] \\
MU_{\mathbb{R}} \arrow[r] & MSpin^c_{\mathbb{R}} \arrow[r] & KU_{\mathbb{R}}
\end{tikzcd}
$$