Towards an equivariant splitting of real spin bordism

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BORDISM RINGS OF MANIFOLDS

$$\Omega_*^G := \left\{ \begin{array}{l} *\text{-dimensional closed} \\ \text{manifolds with } G\text{-structure} \end{array} \right\} \Big/ _{G\text{-bordism}}$$

<u>Classical Problem:</u> Determine Ω_*^G via computable invariants, e.g. characteristic classes/numbers.

ORIENTATIONS OF RING SPECTRA

① $\boxed{\text{Thom classes: } u_V \in E^*(Th(V))}$

↳ Thom isomorphisms in E-cohomology for vector bundles with "G"-structure.

↳ Use presence of G-structures to compute $E^*(\;)$.

② $\boxed{\text{Genera: } \Omega_*^G \longrightarrow E_*}$

↳ Multiplicative bordism invariants for manifolds with "G"-structure.

↳ $\boxed{\text{Use } E_* \text{ to approximate bordism ring } \Omega_*^G.}$

$\boxed{\text{Maps of ring spectra: } MG \longrightarrow E}$

↳ Pontryagin-Thom construction: $MG_* \cong \Omega_*^G$

<u>Example:</u> (Thom, '54)

$MO \longrightarrow H\mathbb{Z}/2 \rightsquigarrow$ unoriented bordism classes are determined by Stiefel-Whitney numbers.

SPIN$^{(c)}$ BORDISM AND TOPOLOGICAL K-THEORY

Atiyah-Bott-Shapiro '63: Constructed orientations

$$MSpin \longrightarrow KO$$

$$MSpin^c \longrightarrow KU$$

Theorem: (Anderson-Brown-Peterson '67)
$[M] \in \Omega_*^{Spin}$ is determined by $\left\{ \begin{array}{l} \text{KO-characteristic numbers \&} \\ \text{Stiefel-Whitney numbers of } M \end{array} \right\}$.

Theorem: (Anderson-Brown-Peterson '67, Stong '68)
$[M] \in \Omega_*^{Spin^c}$ is determined by $\left\{ \begin{array}{l} \text{KU-characteristic numbers \&} \\ \text{Stiefel-Whitney numbers of } M \end{array} \right\}$.

Spin$^{(c)}$ Bordism, K-Theory, and Reality

Orientations with Reality:

$$MSpin \longrightarrow KO = (KU_{\mathbb{R}})^{C_2}$$

$$MU \longrightarrow MSpin^c \longrightarrow KU = (KU_{\mathbb{R}})^e$$

$$MU_{\mathbb{R}} \longrightarrow \boxed{?} \longrightarrow KU_{\mathbb{R}}$$ = Atiyah's "KR"-theory '66

Theorem: (Halladay-K. '24) $\exists$ C_2-E_∞-ring spectrum, $MSpin^c_{\mathbb{R}}$, with

Real spin bordism

(1) $MSpin^c \simeq (MSpin^c_{\mathbb{R}})^e$,

(2) $MSpin \xrightarrow[E_\infty]{} (MSpin^c_{\mathbb{R}})^{C_2}$

(3) $MU_{\mathbb{R}} \xrightarrow[C_2\text{-}E_\infty]{} MSpin^c_{\mathbb{R}} \xrightarrow[C_2\text{-}E_\infty]{} KU_{\mathbb{R}}$

Real spin orientation

REAL SPIN BORDISM — What is it?

$\exists\ C_2 \circlearrowright \mathrm{Spin}^c(p,q) = \mathrm{Spin}^c(p+q)$ such that

- $\mathrm{Spin}(n) \xrightarrow{\sim} \mathrm{Spin}^c(n,0)^{C_2} = \left(\mathrm{Spin}(n) \times_{\{\pm 1\}} U(1)\right)^{C_2}$ $\overline{(\ )}$ on $U(1)$
- $U(n) \longrightarrow \mathrm{Spin}^c(n,n)$.

$\Rightarrow$ [The underlying C_2-action on $M\mathrm{Spin}^c_* \cong \Omega^{\mathrm{Spin}^c}_*$ is "negation of c_1".

[**Theorem:** (Abdallah-Halladay-K. '25) There does not exist a C_2-spectrum, M, with $M^e \simeq M\mathrm{Spin}^c$ & $M^{C_2} \simeq M\mathrm{Spin}$, such that $C_2 \circlearrowright M^e$ induces a ring map on $M\mathrm{Spin}^c_*$.

[**Corollary:** $(M\mathrm{Spin}^c_{\mathbb{R}})^{C_2} \not\simeq M\mathrm{Spin} \not\simeq (M\mathrm{Spin}^c_{\mathbb{R}})^{hC_2}$.

THE ANDERSON-BROWN-PETERSON SPLITTINGS

K-theory characteristic classes: $\forall\, i \in \mathbb{Z}_{\geq 1}$,

$$\begin{tikzcd} \exists\, \pi_r^i \arrow[d, mapsto] & \in & KO^0(BSO) \arrow[d, "\otimes\mathbb{C} = \mathrm{res}"] \\ \pi^i & \in & KU^0(BSO) \end{tikzcd}$$

such that:

- $ch_i(\pi^i) = p_i$,
- $ch_i(\pi^j) = 0$, $i < j$

$$P = \{ I = (i_1, \ldots, i_k) \mid k \geq 0,\ i_\ell \in \mathbb{Z}_{\geq 1} \}$$

Define $\pi_r^I := \pi_r^{i_1} \cdots \pi_r^{i_k} \in KO^0(BSO)$,

& $\pi^I := \pi^{i_1} \cdots \pi^{i_k} \in KU^0(BSO)$.

Theorem: (Anderson-Brown-Peterson '67, Stong '68) $n_I = \begin{cases} 4|I|, & |I| \text{ even} \\ 4|I|-2, & |I| \text{ odd}, \end{cases}$ and $|I| = \sum_{l=1}^{k} i_l$. Then there exist lifts:

$$\begin{tikzcd} & ko\langle n_I\rangle \arrow[d] \\ BSO \arrow[ur, dashed, "\exists"] \arrow[r, "\pi_r^I"'] & KO \end{tikzcd} \quad \& \quad \begin{tikzcd} & ku\langle 4|I|\rangle \arrow[d] \\ BSO \arrow[ur, dashed, "\exists"] \arrow[r, "\pi^I"'] & KU \end{tikzcd}$$

$$\begin{tikzcd} MSpin \arrow[r, "\Delta"] \arrow[rrr, bend left, "f_r^I"] & MSpin \wedge BSpin \arrow[r, "\varphi_r \wedge \pi_r^I"] & ko \wedge ko\langle n_I\rangle \arrow[r, "\mu"] & ko\langle n_I\rangle \end{tikzcd}$$

$$\begin{tikzcd} MSpin^c \arrow[r, "\Delta"] \arrow[rrr, bend left, "f^I"] & MSpin^c \wedge BSpin^c \arrow[r, "\varphi \wedge \pi^I"] & ku \wedge ku\langle 4|I|\rangle \arrow[r, "\mu"] & ku\langle 4|I|\rangle \end{tikzcd}$$

Theorem: (Anderson-Brown-Peterson '67, Stong '68) $\exists\, Z_r \subseteq H^*(MSpin; \mathbb{Z}/2)$ & $Z \subseteq H^*(MSpin^c; \mathbb{Z}/2)$ such that the maps

$$MSpin \xrightarrow{(\vee f_r^I) \vee (\vee z)} \bigvee_{I \in P_1} ko\langle n_I \rangle \vee \bigvee_{z \in Z_r} \Sigma^{|z|} H\mathbb{Z}/2$$

$$\& \quad MSpin^c \xrightarrow{(\vee f^I) \vee (\vee z)} \bigvee_{I \in P} ku\langle 4|I| \rangle \vee \bigvee_{z \in Z} \Sigma^{|z|} H\mathbb{Z}/2$$

are 2-local equivalences.

All torsion is 2-torsion $\Rightarrow$ computes $MSpin_*$ & $MSpin^c_*$

$\Rightarrow$ Classification of $[M] \in \Omega_*^{Spin^{(c)}}$ above.

Goal: Obtain an analogous splitting for $MSpin^c_{\mathbb{R}}$.

EQUIVARIANCE OF THE ANDERSON-BROWN-PETERSON MAP

Theorem: (Abdallah–K. '25) $\exists$ C_2-equivariant map,

$$(MSpin^c_{\mathbb{R}})^e \xrightarrow{(\vee f^I_{\mathbb{R}})\vee(\vee z)} \bigvee_{I\in\mathcal{P}} ku_{\mathbb{R}}\langle 4|I|\rangle^e \vee \bigvee_{z\in Z} \Sigma^{|z|} H\mathbb{Z}/2,$$

in Sp^{BC_2}, whose underlying map is the 2-local splitting of $MSpin^c$ above.

Corollary: (Abdallah–K. '25) $\exists$ 2-local splitting,

$$(MSpin^c_{\mathbb{R}})^{hC_2} \xrightarrow{\simeq_{2\text{-loc}}} \bigvee_{I\in\mathcal{P}} ku_{\mathbb{R}}\langle 4|I|\rangle^{hC_2} \vee \bigvee_{z\in Z} (\Sigma^{|z|} H\mathbb{Z}/2)^{hC_2},$$

of the homotopy fixed points of Real spin bordism.

WHAT ABOUT A GENUINE REFINEMENT?

To refine $\qquad \pi^I : BSpin^c \longrightarrow ku\langle 4|I|\rangle$,

to $\qquad \pi^I : BSpin^c_{\mathbb{R}} \longrightarrow ku_{\mathbb{R}}\langle 4|I|\rangle^e$,

factor $\rightsquigarrow$

$$\begin{tikzcd} BSpin^c_{\mathbb{R}} \arrow[r] \arrow[d] & ku_{\mathbb{R}}\langle 4|I|\rangle^e \\ BSO \arrow[r] & ko\langle n_I\rangle \arrow[u] \end{tikzcd}$$

- $|I|$ even: $\quad n_I = 4|I| \quad \Rightarrow \quad ko\langle 4|I|\rangle \simeq ku_{\mathbb{R}}\langle 4|I|\rangle^{C_2}$

$$\Rightarrow \quad \begin{tikzcd} ko\langle 4|I|\rangle \arrow[r, phantom, "\simeq"] & ku_{\mathbb{R}}\langle 4|I|\rangle^{C_2} \arrow[d] \\ BSO \arrow[u] \arrow[r] & ku_{\mathbb{R}}\langle 4|I|\rangle^e \end{tikzcd}$$

- $|I|$ odd: $\quad n_I = 4|I|-2$

$$\Rightarrow \quad \begin{tikzcd} & ko\langle 4|I|\rangle \arrow[r] \arrow[d] & ku_{\mathbb{R}}\langle 4|I|\rangle^e \arrow[d] \\ BSO \arrow[ur, dotted, "\nexists"] \arrow[r] & ko\langle 4|I|-2\rangle \arrow[r] \arrow[ur, "\exists c"] & ku_{\mathbb{R}}\langle 4|I|-2\rangle^e \end{tikzcd}$$.

Proposition: $\exists$ a genuine C_2-spectrum,

$$ku_{\mathbb{R}}\langle 4n,2\rangle \simeq \left(\begin{tikzcd} ko\langle 4n-2\rangle \arrow[d, shift right, "c"'] & \\ ku\langle 4n\rangle \arrow[u, shift right] \arrow[loop below, "\overline{(\ )}"] \arrow[r] & ko\langle 4n\rangle \arrow[ul] \end{tikzcd}\right).$$

Pf:

$$\begin{tikzcd} ku_{\mathbb{R}}\langle 4n\rangle_{hC_2} \arrow[r] \arrow[d, equal] & ko\langle 4n\rangle \arrow[rr] \arrow[dr] & & ko\langle 4n-2\rangle \arrow[r] \arrow[dl] \arrow[d, phantom, "\circlearrowleft"] & ku_{\mathbb{R}}\langle 4n,2\rangle^{gC_2} \arrow[d] \arrow[dl, phantom, "\ulcorner"] \\ ku_{\mathbb{R}}\langle 4n\rangle_{hC_2} \arrow[rr] & & ku_{\mathbb{R}}\langle 4n\rangle^{hC_2} \arrow[rr] & & ku_{\mathbb{R}}\langle 4n\rangle^{tC_2} \end{tikzcd}.$$

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Proposition: The map $\pi^I : BSO \longrightarrow ku\langle 4|I|\rangle$ refines to a genuine equivariant map $BSO \longrightarrow ku_{\mathbb{R}}\langle 4|I|, 2\rangle$ ($|I|$ odd).

FUTURE WORK

① Does the Anderson-Brown-Peterson map refine to a genuine equivariant map:

$$MSpin^c_{\mathbb{R}} \longrightarrow \bigvee_{I \in P_{even}} ku_{\mathbb{R}}\langle 4|I|\rangle \vee \bigvee_{I \in P_{odd}} ku_{\mathbb{R}}\langle 4|I|,2\rangle \vee X \ ?$$

② If so, how close is it to being an equivalence?

THANK YOU FOR LISTENING!