REAL SPIN BORDISM & ORIENTATIONS OF TOPOLOGICAL K-THEORY

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joint with Zach Halladay

Classical Orientations & K-theory

Start with the type of orientation all spectra have

E = cohomology theory

Eilenberg-Steenrod axioms

$\Rightarrow$ [Suspension isomorphism:

$$E^n(X) \cong E^{n+1}(S^1 \wedge X)$$

"$\mathbb{S}$-module"

In terms of spectra: structure maps $S^1 \wedge E_n \xrightarrow{\sigma} E_{n+1}$

$\alpha \in E^n(X)$:

$$S^1 \wedge X \xrightarrow{1 \wedge \alpha} S^1 \wedge E_n \xrightarrow{\sigma} E_{n+1}$$

E = multiplicative / ring spectrum

$\Rightarrow$
- $E_p \wedge E_q \xrightarrow{m} E_{p+q}$ mult.
- $S^n \xrightarrow[\eta]{} E_n$ unit

$\Big\}$ "$\mathbb{S}$-algebra"

↳ multiplicative

$$\Rightarrow \quad \begin{tikzcd} S^1 \wedge E_n \arrow[r, "\eta \wedge 1"] \arrow[rr, bend right, "\sigma"'] & E_1 \wedge E_n \arrow[r, "m"] & E_{n+1} \end{tikzcd}$$

$\Rightarrow$ suspension isomorphism is multiplication by a <u>suspension class</u> $\eta_1 \in E^1(S^1)$.

$$\boxed{\text{orientation} = \text{generalization of suspension class / isomorphism}}$$

$$\begin{tikzcd} \mathbb{S} \arrow[r, "E_\infty"] \arrow[d, equal] & E \\ MFr \arrow[d] & \\ MX \arrow[uur, dashed, "\exists \text{ ring map? } E_\infty ?"'] \arrow[d, dotted] & \\ MO & \end{tikzcd}$$

<u>genus</u> $\approx$ integration of E-cohomology classes on manifolds.

$$S^1 \wedge X = \mathrm{Thom}(\mathbb{R}^1 \times X)$$

"suspension class" for nontrivial (twisted) vector bundles over X?

↳ <u>Thom class</u>

Examples in ordinary cohomology

- $H\mathbb{Z}/2$: $V \to X$ any vector bundle rank n

  $\Rightarrow$ $H^*(X;\mathbb{Z}/2) \xrightarrow[\cdot\tau]{\cong} H^{*+n}(V;\mathbb{Z}/2)$ Thom isomorphism

  $V = EO(n)\times_{O(n)}\mathbb{R}^n \to BO(n)$ universal rank n v.b.

  $\Rightarrow$ Thom class $\tau_n \in H^n(MO(n);\mathbb{Z}/2)$

  $\Leftrightarrow$ $\tau_n : MO(n) \to (H\mathbb{Z}/2)_n = K(\mathbb{Z}/2, n)$

  $\Rightarrow$ $MO \to H\mathbb{Z}/2$ "$\mathbb{Z}/2$ fundamental class"

- $H\mathbb{Z}$: $V \to X$ oriented vector bundle

  $\Rightarrow$ Thom isomorphism $\Rightarrow$ $MSO \to H\mathbb{Z} \to H\mathbb{R}$ "fundamental class".

K-theory: KU, KO → don't have Thom isomorphisms for all oriented vector bundles.

- [Atiyah-Hirzebruch, Bott, ~1960]

$$MU \longrightarrow KU$$ "Todd genus"

by constructing Thom classes for complex vector bundles via exterior bundles.

- [Atiyah-Bott-Shapiro, '63]

$$MSpin \longrightarrow KO$$ "$\hat{A}$-genus"

$$MU \longrightarrow MSpin^c \longrightarrow KU$$ "Todd genus"

by constructing Thom classes via Clifford modules.

- [Atiyah, '66] "Real" K-theory, $KR = KU_{\mathbb{R}}$, a C_2-equivariant cohomology theory.

Def: A Real object in some category $\mathcal{C}$ is an object $a \in \mathcal{C}$ together with $C_2 \to \mathrm{Aut}_{\mathcal{C}}(a)$. A Real morphism is a morphism of Real objects that commutes with the Real structures.

Why Real? Fix $\mathbb{C}$ to be the complex numbers with complex conjugation.

⇒'ed Def: A Real vector space (vector space over $\mathbb{C}$) is a complex vector space with a complex conjugate linear involution.

↳ similarly Real vector bundle over a (C_2-) Real space.

[**Def:** $KU_{\mathbb{R}}(X)$ = Groth. gp. of Real v.b.'s on X.

↳ leads to a C_2-equivariant cohomology theory $KU_{\mathbb{R}}$.

↳ $KU_{\mathbb{R}}^{C_2} \simeq KO$, $KU_{\mathbb{R}}^{\{e\}} \simeq KU$

[**Thm:** (Atiyah, '66): Real vector bundles have Thom classes in $KU_{\mathbb{R}}$-cohomology.

↳ Thom spectrum of Real vector bundles?

$EU(n) \times_{U(n)} \mathbb{C}^n \longrightarrow BU(n)$ universal complex bundle

$\circlearrowright$ C_2 $\qquad$ $\circlearrowright$ C_2 $\qquad$ natural Real structure.

$\Rightarrow$ universal Real v.b.

$\Rightarrow$ $C_2 \circlearrowright MU(n)$ $\Rightarrow$ C_2-spectrum $MU_{\mathbb{R}}$ = Real bordism

[Landweber, late '60s; Araki, late 70's; Hu-Kriz, ~2000]

& $\exists$ ring map $MU_{\mathbb{R}} \longrightarrow KU_{\mathbb{R}}$.

K-theory orientations

$$MSpin \to KO \simeq KU_{\mathbb{R}}^{C_2}$$

E_∞: Joachim

$$MU \longrightarrow MSpin^c \longrightarrow KU \simeq KU_{\mathbb{R}}^{\{e\}}$$

$$MU_{\mathbb{R}} \longrightarrow \boxed{?} \longrightarrow KU_{\mathbb{R}}$$

E_∞: Hu-Kriz

Further, all known orientations are E_∞, so want $\boxed{?}$ factor through E_∞-maps.

Evidence: [Atiyah, '67] idea: Real version of $spin^c$.

$\mathbb{R}^{p,q}$ = C_2-representation

$\Rightarrow C_2 \circlearrowright Cl(\mathbb{R}^{p,q})$ via universal property

$\Rightarrow C_2 \circlearrowright \mathbb{C}l(\mathbb{R}^{p,q})$ extending $\uparrow$ via complex conj.

UI

comment relation to ABS isomorphism with $\pi_\star KU_{\mathbb{R}}$.

$\Rightarrow C_2 \circlearrowright \mathrm{Spin}^c(p,q)$ "Real group"

Thm: (Atiyah) $G \to \mathrm{Spin}^c(p,q)$ Real homomorphism

$$\Rightarrow \quad KU_{\mathbb{R},G}(X) \xrightarrow[\cdot \tau]{\sim} KU_{\mathbb{R},G}(\mathbb{R}^{p,q} \times X), \quad p \equiv q \bmod 8.$$

↳ "equivariant Bott periodicity"

↳ contains all K-thy Thom isom's as special cases.

e.g. • $\mathrm{Spin}(8n) \hookrightarrow \mathrm{Spin}^c(8n,0)$
with trivial C_2-action

$$\begin{array}{cccc} \Rightarrow & KO_{\mathrm{Spin}(8n)}(P) & \xrightarrow{\sim} & KO_{\mathrm{Spin}(8n)}(\mathbb{R}^{8n} \times P) \\ & \| & & \| \\ & KO(B) & \simeq & KO(\mathbb{R}^{8n} \times_{\mathrm{Spin}(8n)} P) \end{array}$$

• $U(n) \hookrightarrow \mathrm{Spin}^c(n,n)$ C_2-equivariant.

Main Thm : (Halladay-K) $\exists$ C_2-E_∞-ring spectrum $MSpin^c_{\mathbb{R}}$ (Real spin bordism), s.t.

(1) $MSpin^c \simeq (MSpin^c_{\mathbb{R}})^{\{e\}}$

(2) $\exists$ E_∞-map $MSpin \longrightarrow (MSpin^c_{\mathbb{R}})^{C_2}$

(3) $\exists$ C_2-E_∞-maps $MU_{\mathbb{R}} \xrightarrow{(3a)} MSpin^c_{\mathbb{R}} \xrightarrow{(3b)} KU_{\mathbb{R}}$.

$\boxed{\text{Real spin orientation}}$

↳ Recovers other orientations

$(1) + (3) \Rightarrow MSpin^c \longrightarrow KU$

$(2) + (3) \Rightarrow MSpin \longrightarrow KO$

compose $(3) \Rightarrow MU_{\mathbb{R}} \longrightarrow KU_{\mathbb{R}}$

Picture on side board:

$$\begin{tikzcd}
 & MSpin \arrow[r] \arrow[d] & KO \arrow[d, "\wr"] \\
 & (MSpin^c_{\mathbb{R}})^{C_2} \arrow[r] \arrow[d] & (KU_{\mathbb{R}})^{C_2} \arrow[d] \\
MU_{\mathbb{R}} \arrow[r] & MSpin^c_{\mathbb{R}} \arrow[r] & KU_{\mathbb{R}}
\end{tikzcd}$$

Proof idea

- construction + (3b): Follow Joachim & equip everything with Real structures.

$$\left(C_2\text{-}E_\infty\text{-ring spectrum}\right) \Longleftrightarrow \left(\begin{array}{c}\text{commutative monoid in} \\ \text{orthogonal } C_2\text{-spectra}\end{array}\right)$$

$\hookrightarrow \left[\mathbb{R}^{p,q} \mapsto E(\mathbb{R}^{p,q}) \ C_2\text{-space}\right]$

$$ESpin^c(p,q) = \mathcal{U}^{ev}\left(Cl(\mathbb{R}^{p,q}) \otimes L^2(\mathbb{R}^{p,q})\right)$$

$$\Rightarrow \quad MSpin^c(p,q) = Thom\left(ESpin^c(p,q) \times_{Spin^c(p,q)} \mathbb{R}^{p,q}\right)$$

$$KU_{\mathbb{R}}(p,q) = C^*_{gr}\left(S^1, Cl(\mathbb{R}^{p,q}) \otimes \mathcal{K}_{\mathbb{R}^{p,q}}\right)$$

Kasparov: originally developed KK with Real structures.

$\hookrightarrow$ along w/ Bott periodicity $\sim$ Atiyah's.

- (1) + (2): Follow directly from the construction & Kuiper.

- (3a) return if time.

Fixed points of $MSpin^c_{\mathbb{R}}$

Recall:

$$\begin{tikzcd} Spin(n) \arrow[rr] \arrow[dr, "\wr"'] & & Spin^c(n) \\ & Spin^c(n)^{C_2} \arrow[ur, hook] & \end{tikzcd}$$

(2)

$$\begin{tikzcd} MSpin \arrow[rr, "u"] \arrow[dr, "w"'] & & MSpin^c \\ & (MSpin^c_{\mathbb{R}})^{C_2} \arrow[ur, "res"'] & \end{tikzcd}$$

<u>Thm</u>: (Halladay - K) Any w factoring u is not an equivalence. (G)

<u>Thm</u>: (Abdallah - Halladay - K)

$$MSpin \not\simeq (MSpin^c_{\mathbb{R}})^{C_2}$$

uses more specific algebraic knowledge of rings $MSpin_*$, $MSpin^c_*$.

(A)

proof (G): $MSpin^c_{\mathbb{R}}$ = (genuine) C_2-spectrum

$\Rightarrow$ homotopy Mackey functor $\underline{\pi}_* MSpin^c_{\mathbb{R}}$ has data:

assume w is an equiv.

$$\begin{array}{ccc} \pi_*^{C_2} MSpin^c_{\mathbb{R}} & \underset{tr}{\overset{res}{\rightleftarrows}} & \pi_*^{e} MSpin^c_{\mathbb{R}} \circlearrowleft C_2 \\ \wr\!\wr & & \wr\!\wr \\ MSpin_* & \underset{tr}{\overset{u}{\rightleftarrows}} & MSpin^c_* \circlearrowleft C_2 = \overline{(\cdot)} \end{array}$$

[property of Mackey functors: $\alpha + \bar{\alpha} = u(tr(\alpha))$.

the map that views a spin^c manifold as a spin manifold

$\Rightarrow$ $\alpha + \bar{\alpha} \in MSpin^c_*$ can be rep'd by a spin manifold

$\Rightarrow$ $\boxed{\hat{A}(\alpha + \bar{\alpha}) \in \mathbb{Z}, \quad \forall\, \alpha \in MSpin^c_* \qquad (*)}$

$[\mathbb{CP}^2] \in MSpin^c_4$ w/ $\hat{A}([\mathbb{CP}^2]) = -\frac{1}{8}$

$\hat{A}([\mathbb{CP}^2] + [\overline{\mathbb{CP}^2}]) = n \in \mathbb{Z} \Rightarrow \hat{A}([\overline{\mathbb{CP}^2}]) = \frac{1}{8} + n$

$MSpin^c_{\mathbb{R}}$ a C_2-ring spectrum $\Rightarrow$ $(\bar{\cdot})$ a ring map.

Consider $[\mathbb{CP}^2] \in MSpin^c_8$.

$$\hat{A}([\mathbb{CP}^2]^2 + [\overline{\mathbb{CP}^2}]^2) = \frac{8n+1}{32} + n^2 \notin \mathbb{Z} \quad ※$$

<u>Remarks</u>: ① Proof does not depend on specific $C_2 \curvearrowright MSpin^c_{\mathbb{R}}$. (also for (A).) So

Thm: $\nexists$ C_2-ring spectrum with underlying ring spectrum $MSpin^c$ & fixed points $MSpin$.

② $MSpin^c_{\mathbb{R}}$ constructed as a genuine C_2-spectrum, but $C_2 \curvearrowright Spin^c(n) \Rightarrow C_2 \curvearrowright MSpin^c \Rightarrow (MSpin^c_{\mathbb{R}})^h$.

Our results apply to both $MSpin^c_{\mathbb{R}}$ & $\lrcorner$.

<u>Q</u>: $MSpin^c_{\mathbb{R}} \simeq (MSpin^c_{\mathbb{R}})^h$?

<u>Proof of (3a)</u>: $MU_{\mathbb{R}} \longrightarrow MSpin^c_{\mathbb{R}}$.

"<u>Reason it's true</u>": $U(n) \hookrightarrow Spin^c(n,n)$ is C_2-equivariant.

free ↻ ⇐ ↺ free

Then given

$$ESpin^c(n,n)$$

$\Rightarrow$ C_2-spectrum $\widetilde{MU}_{\mathbb{R}}$ with underlying MU

& $\widetilde{MU}_{\mathbb{R}} \to MSpin^c_{\mathbb{R}}$. Just need $\widetilde{MU}_{\mathbb{R}} \simeq MU_{\mathbb{R}}$.

Follows from ↘

<u>Lemma</u>: (Halladay-K) X, Y Real $U(n)$-spaces with $X \circlearrowleft U(n)$ freely, then

$$(X \times_{U(n)} Y)^{C_2} \cong X^{C_2} \times_{O(n)} Y^{C_2}.$$

Proof uses that symmetric unitary matrices have symmetric unitary square roots.

(<u>BTW</u>: $U(n)$ is an "algebraically closed" group!)

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