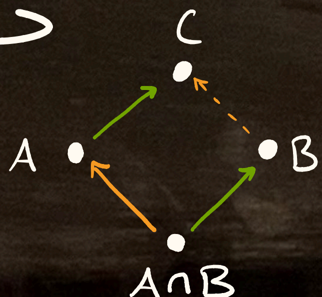
TRANSFER SYSTEMS IN EQUIVARIANT HOMOTOPY THEORY

& compatible pairs



{Backgroud for ongoing work jt. with DeMark, Hill, Niu, Stoeckl, Van Niel, & Yan}

Transfer Systems

Def: Let P be a small category. A *transfer system* on P is a wide subcategory of P that is closed under pullbacks.

Case of interest: $P = (P, \leq)$ is a poset.

Then a transfer system on $(P, \leq) \iff$ a partial order $\longrightarrow$ on P such that

(1) $a \longrightarrow b \implies a \leq b$

(2)
$$\begin{tikzcd}
 & c & \\
a \arrow[ur] & & b \arrow[ul, phantom, "\leq" sloped] \\
 & x \arrow[ul, phantom, "\leq" sloped] \arrow[ur, phantom, "\leq" sloped, near start] \arrow[ur] &
\end{tikzcd}$$
& x maximal w.r.t. this property $\implies$
$$\begin{tikzcd}
 & b \\
x \arrow[ur] &
\end{tikzcd}$$

Case of interest: $(P, \leq)$ is a lattice, i.e. a $(0,1)$-category w/ finite limits (meets, $\wedge$) & colimits (joins, $\vee$).

$\hookrightarrow$ Thm $(2) \iff (2')$

(2')

$$\begin{tikzcd}
 & c & \\
a \arrow[ur] & & b \arrow[ul, phantom, sloped, "\leq"] \\
 & a \wedge b \arrow[ul, phantom, sloped, "\leq"] \arrow[ur] &
\end{tikzcd}
\quad \Longrightarrow \quad a \wedge b \to b$$

<u>Ex:</u> If $a, b, c \in P$ modeled by sets, think of $a \wedge b \to b$ as restriction of $a \longrightarrow c$ to b.

<u>Note</u>: If L is a lattice, then

$$\mathrm{Tr}\,L = \{\text{transfer systems on } L\}$$

is also a lattice, where if $\xrightarrow[1]{}, \xrightarrow[2]{} \in \mathrm{Tr}\,L$,

$$(\xrightarrow[1]{}) \leq (\xrightarrow[2]{}) \iff (a \xrightarrow[1]{} b \Rightarrow a \xrightarrow[2]{} b).$$

[<u>Problem</u>: Given L, determine $\mathrm{Tr}\,L$.

↳ HARD!

<u>Ex</u>: $L = [n] = 0 \leq 1 \leq \cdots \leq n$.

$$|\mathrm{Tr}[n]| = \frac{1}{2n+3}\binom{2n+3}{n+1} = (n+1)^{\text{th}} \text{ Catalan number.}$$

For equivariant homotopy theory, $L = \mathrm{Sub}\, G$, but we need a slightly stronger notion.

Def: A G-transfer system is a transfer system on $\mathrm{Sub}\, G$ that is closed under conjugation, i.e.

$$(a \longrightarrow b) \Rightarrow (gag^{-1} \longrightarrow gbg^{-1} \quad \forall g \in G).$$

$\mathrm{Tr}\, G :=$ lattice of G-transfer systems.

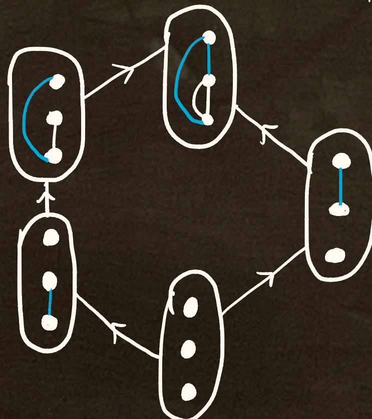
Ex: G abelian $\Rightarrow \mathrm{Tr}\, G = \mathrm{Tr}(\mathrm{Sub}\, G)$.

Ex: p prime, $n \geq 1$. $\mathrm{Sub}(C_{p^n}) \cong [n]$.

$|\mathrm{Tr}\, C_{p^n}| \rightsquigarrow$ Catalan numbers above.

$\underline{n=2}$: $\quad \mathrm{Sub}(C_{p^2}) = \begin{array}{l} \bullet\ \langle 1\rangle \\ \rotatebox{90}{$\leq$} \\ \bullet\ \langle p\rangle \\ \rotatebox{90}{$\leq$} \\ \bullet\ \langle p^2\rangle \end{array}$, $\quad \dfrac{1}{2\cdot 2+3}\dbinom{2\cdot 2+3}{2+1} = \dfrac{6\cdot 5\cdot 4}{4\cdot 3\cdot 2}$

$= 5.$



side board & keep up.

More generally, $\mathrm{Sub}(C_{p_1^{n_1}\cdots p_k^{n_k}})$ is a k-dim'l rectangular grid with side lengths $n_1, \ldots, n_k$.

$\mathrm{Tr}(C_{p_1^{n_1}\cdots p_k^{n_k}})$ is very complicated!

N_∞-operads

Def: A *G-operad* $\{\mathcal{O}(n)\}_{n\geq 0}$ consists of

- $\forall n \geq 0$, a $G\times\Sigma_n$-space $\mathcal{O}(n)$.
- $1 \in \mathcal{O}(1)^G$.
- G-equivariant
$$\mathcal{O}(k)\times\mathcal{O}(n_1)\times\cdots\times\mathcal{O}(n_k) \to \mathcal{O}(n_1+\cdots+n_k)$$

s.t. ... "coherence" ...

Def: A G-operad $\{\mathcal{O}(n)\}_{n\geq 0}$ is an *N_∞-operad* if

- $\mathcal{O}(0) \simeq *$.

- $\Sigma_n \circlearrowright \mathcal{O}(n)$ is free
- $\forall\ H \leq G \times \Sigma_n$, $\mathcal{O}(n)^H \in \{*, \emptyset\}$.
- $\{H \leq G \times \Sigma_n \mid \mathcal{O}(n)^H \simeq *\}$ is closed under conjugacy, taking subgroups, & contains all $K \times \{e\}$.

$G = * \Rightarrow E_\infty$-operad $\rightsquigarrow$ unique!

<u>Def</u>: A <u>map</u> of G-operads $\mathcal{O}_1 \to \mathcal{O}_2$ is a sequence $\{\mathcal{O}_1(n) \to \mathcal{O}_2(n)\}$ compatible w/ the str. maps.

$G \times \Sigma_n$-equivariant

A map $\mathcal{O}_1 \to \mathcal{O}_2$ is a <u>weak equivalence</u> if $\mathcal{O}_1(n)^H \longrightarrow \mathcal{O}_1(n)^H$ is a w.e. $\forall H \leq G \times \Sigma_n$, $\forall n \geq 0$.

Thm: (Blumberg–Hill, Gutiérrez–White, Bonventre–Pereira, Rubin, Balchin–Barnes–Roitzheim)

Let $N_\infty\text{-}Op^G$ be the cat. of N_∞-operads & $W \subseteq N_\infty\text{-}Op^G$ the weak equivalences. Then there is an equivalence of categories:

$$h\left(N_\infty\text{-}Op^G[W^{-1}]\right) \simeq \mathrm{Tr}\, G$$

complicated!

The map: $\mathcal{O} \longmapsto \left(\xrightarrow[\mathcal{O}]{} \right)$ where

$$K \xrightarrow[\mathcal{O}]{} H \quad \text{iff} \quad \mathcal{O}([H:K])^{\Gamma(H/K)} \simeq *,$$

where $\Gamma(H/K) \leq G \times \Sigma_{[H:K]}$ is the graph of some $\sigma : H \to \Sigma_{[H:K]}$ $\Leftrightarrow$ "$\mathcal{O}$ has H/K-norms".

$\mathcal{O} \in N_\infty\text{-}\mathrm{Op}^G$ & $A = \mathcal{O}$-algebra.

A = "E_∞-algebra + $\underbrace{H/K\text{-norms}}$ for $K \xrightarrow[\mathcal{O}]{} H$."

For $K \leq H$, always have

$$\begin{tikzcd} A^H \arrow[r, hook] & A^K \arrow[l, bend left, "H/K\text{-norm}"] \end{tikzcd}$$

H/K-norm (+ coherence).

For applications of presence of norms:

See Hill-Hopkins-Ravenel,

"On the non-existence of elements of Kervaire invariant one".

<u>Note</u>: In additive contexts, norms are called "transfers". Since all G-spectra* are additively N_∞-algebras, always have "transfers".

$\downarrow$

(Mackey functors)

* indexed on a complete universe.

Examples: Let $\mathcal{U}$ be a **G-universe**: ∞-dim'l G-rep s.t. each subrep appears ∞'ly often.

(1) **Little disks***: $V \subseteq \mathcal{U}$ finite dim'l.

$$D(V)_n = \{ \textstyle\coprod_n D(V) \hookrightarrow D(V) \}$$

$$D(\mathcal{U})(n) = \operatorname{colim}_{V \subseteq \mathcal{U}} D(V)_n .$$

* Really, use the Steiner operad instead, due to issues w/ products.
Idea: uses the whole space V instead of D(V).

(2) **Linear isometries**: $\mathcal{L}(\mathcal{U})(n) = \mathcal{L}(\mathcal{U}^n, \mathcal{U})$

space of (nonequivariant) linear isometries $\mathcal{U}^n \to \mathcal{U}$

Thm: (BH) $\forall$ G-universe $\mathcal{U}$, $D(\mathcal{U})$ & $\mathcal{L}(\mathcal{U})$ are N_∞-operads.

Ex: $\mathcal{U} = \bigoplus_{V \text{ irrep}} V^{\oplus \infty}$ **complete G-universe**

$\Rightarrow D(\mathcal{U}) \simeq \mathcal{L}(\mathcal{U}) =:$ **complete N_∞-operad**.

$\approx$ transfer system $=$ Sub G.

Thm: (BH) $\forall G \neq *, C_2, C_3$ $\exists$ G-universe $\mathcal{U}$ s.t. $\mathcal{D}(\mathcal{U}) \neq \mathcal{L}(\mathcal{U})$.

Ex: $G = C_p$, $p \geq 5$. Let λ be a non-trivial C_p-irrep, e.g. $C_p \subseteq U(1) \subseteq \mathbb{C}$

$$\Rightarrow \quad g \cdot z = g^k z.$$

Then let $\mathcal{U} = \mathbb{R}^{\oplus \infty} \oplus \lambda^{\oplus \infty}$.

$$\Rightarrow \quad \mathcal{D}(\mathcal{U}) = \bullet \longrightarrow \bullet \qquad \text{(complete)}$$

$$\mathcal{L}(\mathcal{U}) = \bullet \quad \bullet \qquad \text{(trivial)}$$

In fact: (BH) $\forall G \, \forall \mathcal{U}$, $\exists$ map $\mathcal{L}(\mathcal{U}) \to \mathcal{D}(\mathcal{U}) \in N_\infty\text{-}Op^G$.

i.e. as transfer systems, $\mathcal{L}(\mathcal{U}) \leq \mathcal{D}(\mathcal{U})$.

Remark: • $X \in Sp_G$ = orthogonal spectra indexed on $\mathcal{U}$.

$\Rightarrow$ additive structure of X arises from a $D(\mathcal{U})$-algebra structure on X.

$\Rightarrow$ $\underline{\pi}_* X$ has transfers that $D(\mathcal{U})$ has.

• $R \in Sp_G$ comm. monoid object $\Rightarrow$ multiplicative structure on R $\approx$ $\mathcal{L}(\mathcal{U})$-algebra structure.

$\underline{\pi}_0 R$ has norms that $\mathcal{L}(\mathcal{U})$ has.

Q: Why do we care that $h(N_\infty\text{-}Op^G) \simeq \mathrm{Tr}\,G$ is complicated? Why not always use complete?

A: Bousfield localization does not preserve N_∞-structures (i.e. doesn't preserve norms)!

↳ In contrast, E_∞-structures are preserved.

<u>Q:</u> $R \in Sp_G$ comm. monoid $\Rightarrow$ R is both a $\mathcal{D}(\mathcal{U})$-algebra & an $\mathcal{L}(\mathcal{U})$-algebra, in a compatible way. Under what conditions can $(\mathcal{O}_a, \mathcal{O}_m)$ encode the two operations of a ring?

encode key properties of $\mathcal{D}(\mathcal{U})$ & $\mathcal{L}(\mathcal{U})$ in terms of transfer systems

compatibility ?

$\approx$ <u>polynomials</u>

$$\downarrow$$

$$T \xleftarrow[\text{picking variables}]{} A \xrightarrow[\text{multiply}]{} B \xrightarrow[\text{add}]{} S$$

$\to$ want to be able to compose polynomials.

Def/Thm: Given $(\mathcal{O}_a, \mathcal{O}_m)$, there is a well-defined category of polynomials iff $\mathcal{O}_m \leq \mathcal{O}_a$ &

$\forall$

$$\begin{tikzcd}
 & C & \\
A \arrow[ur, "\mathcal{O}_m"] & & B \arrow[ul, dashed, "\exists \mathcal{O}_a"'] \\
 & A \cap B \arrow[ul, "\mathcal{O}_a"] \arrow[ur, "\mathcal{O}_m"'] &
\end{tikzcd}$$

in Sub G

call $(\mathcal{O}_a, \mathcal{O}_m)$ a <u>compatible pair</u>.

Def: • $T \in \mathrm{Tr}\, G$ is <u>disk-like</u> (or <u>cosaturated</u>) if $\forall\, A \to B \in T$ is restriction of some $C \to G$.

• $T \in \mathrm{Tr}\, G$ is saturated if $\forall\, A \to B \in T$ $[A, B] \subseteq T$. e.g.

$$\begin{tikzcd}
 & B & \\
A \arrow[ur] \arrow[rr, "\text{rest.}"'] & & C \arrow[ul, "\text{sat.}"']
\end{tikzcd}$$

Thm: $\mathcal{U}$ = a G-universe. Then $D(\mathcal{U})$ is disk-like & $\mathcal{L}(\mathcal{U})$ is saturated.

↳ converse not quite true.

Q: Given $\mathcal{O}_a$ disk-like, what is the maximal $\mathcal{O}_m \leq \mathcal{O}_a$ s.t. $(\mathcal{O}_a, \mathcal{O}_m)$ is compatible?

ongoing work!

Ex: $G = C_{p^2}$

- disk-like
- saturated
- maximal compatible pair