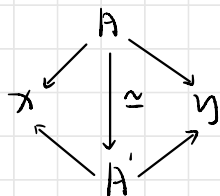
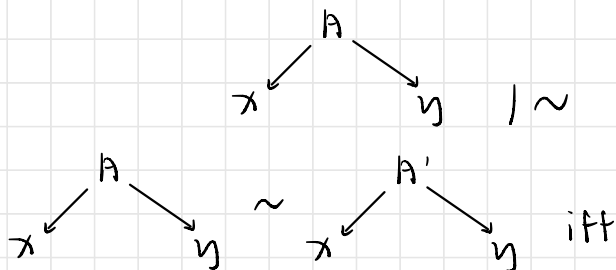


Equivariant algebra & $\mathbb{R}(G)$ -graded cohomology

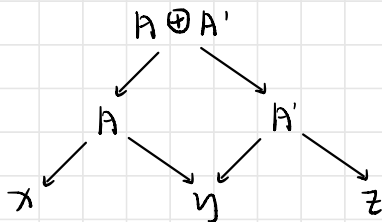
Equivariant algebra

DEF (Span) Given a category $\mathcal{C}$,
 $\text{Span}(\mathcal{C})$: ob: $\text{ob}(\mathcal{C})$

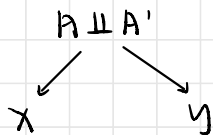
mor: $\text{Hom}_{\text{Span}(\mathcal{C})}(x, y)$



Composition



Disjoint union



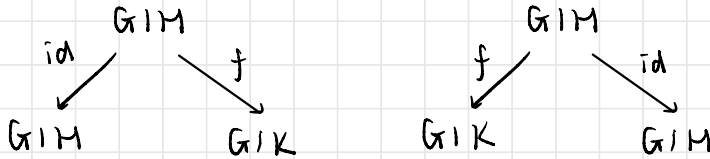
DEF (Burnside category)

$$\mathcal{B}G := \text{Span}^+(G\text{-set fin})$$

DEF (Mackey functor) A Mackey functor is a contravariant functor

$$B_G^{op} \longrightarrow Ab$$

Given $GIM, GIK \in \mathcal{V}_G$. $f: GIM \longrightarrow GIK$



$$T: \mathcal{V}_G^{op} \longrightarrow B_G^{op} \longrightarrow Ab \quad R: \mathcal{V}_G \longrightarrow B_G^{op} \longrightarrow Ab$$

Observations $GIM \xrightarrow{f} GIK \xrightarrow{g} GIJ$

① $T(fg) = T(f)T(g)$

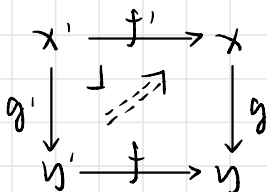
② $R(fg) = R(g)R(f)$

③ Span



$$= T(g)R(f) = \begin{array}{ccccc} & & GIK & & \\ & \swarrow & & \searrow & \\ GIM & & GIK & & GIJ \\ & \swarrow & \searrow & \swarrow & \searrow \\ & GIM & GIK & & GIJ \end{array}$$

④ Beck-Chevalley condition:



$$R(g)T(f) = T(f')R(g')$$

motivates an alternative def of Mackey functor w/o the use of span

DEF (Mackey functor 2)

$$\begin{cases} M_* : \mathcal{U}_G \longrightarrow \mathcal{A}b \\ M^* : \mathcal{U}_G^{op} \longrightarrow \mathcal{A}b \end{cases}$$

① $M_*(X) = M^*(X)$

② Given

$$\begin{array}{ccc} x' & \xrightarrow{f'} & x \\ g' \downarrow & \lrcorner & \downarrow g \\ y' & \xrightarrow{f} & y \end{array} \quad \begin{aligned} M^*(g) M_*(f) \\ = M_*(f') M^*(g') \end{aligned}$$

③ For $X \longrightarrow X \amalg Y \longleftarrow Y$

$$M_*(X) \oplus M_*(Y) = M_*(X \amalg Y)$$

DEF (Mackey functor)

A function $M: \mathcal{V}_G \longrightarrow \text{Ab}$ with

$$\begin{cases} R_K^H: M(H) \rightarrow M(K) & G \text{ abelian} \\ T_K^H: M(K) \rightarrow M(H) & K \subseteq H \\ c_g: M(H) \rightarrow M(gHg^{-1}), g \in G & G/K \rightarrow G/H \end{cases}$$

① $R_H^H, T_H^H, c_h: M(H) \rightarrow M(H)$

are identity for $h \in H$.

② $R_H^J = R_H^K R_K^J, T_H^J = T_K^J T_H^K, H \subseteq K \subseteq J$

③ $c_g c_{g'} = c_g c_{g'}$

④ $c_g R_H^K = R_{gHg^{-1}}^{gKg^{-1}} c_g, c_g T_H^K = T_{gHg^{-1}}^{gKg^{-1}} c_g$

⑤ $R_K^J T_H^J$
 $= \bigoplus_{x \in K \backslash J/H} T_{K \cap xHx^{-1}}^K c_x R_{H \cap x^{-1}Kx}^H$
 $H, K \subseteq J$

Axiomatic
def of Mackey
functor

not abelian

$$K \subseteq xHx^{-1}$$

$$\begin{array}{c} \Downarrow J(H \cap xKx^{-1}) \\ x \in [H \backslash J/K] \\ \parallel \end{array}$$

$$\begin{array}{ccc} J/H \times J/K & \rightarrow & J/H \\ \downarrow \lrcorner & & \downarrow \\ J/K & \longrightarrow & J/J \end{array}$$

EX (Mackey functor)

• Constant $\mathbb{Z}: R_K^H \text{ id } T_K^H \times |H/K|$

• $\pi_n(X): G/H \longrightarrow \pi_n^H(X)$

• Burnside Mackey functor A_G

$$A_G(G/H) = \{ \text{MSet fin}, \mathbb{Z} \}^+$$

R_K^H : restriction along $K \longrightarrow H$

$T_K^H: H \times_K -$

$$K \subseteq xHx^{-1}$$

- Representation R_G

$$R_G(G/H) = \{ \text{fin-dim } H\text{-rep} \}^+$$

$$R_K^H: \text{restriction along } K \longrightarrow H$$

$$T_K^H: \mathbb{Z}[H] \times \mathbb{Z}[K] \longrightarrow$$

- Cohomological Mackey functor:

$$\text{if } T_K^H R_K^H: M(H) \longrightarrow M(K) \text{ is } \times |H/K|$$

- constant
- group cohomology $H^n(G, M)$
- Tate cohomology $\tilde{H}^n(G, M)$
- $G = \text{Gal}(L/K)$, $H \subseteq G$
Let $M_G(G/H) = \mathcal{V}_{L/H}$

- For a set $S \subseteq G$

$$\underline{M}_G(G/H) = \text{Map}^G(G/H, \mathbb{Z}[S]) = \mathbb{Z}[S]^H$$

$$\underline{M}_G(X \xleftarrow{p} A \xrightarrow{q} Y): \underline{M}(X) \longrightarrow \underline{M}(Y)$$

$$s \mapsto (y) \mapsto \sum_{q(a)=y} s(p(a))$$

$$\text{Mack}_G \xrightleftharpoons[\underline{M}_G]{\omega} \text{GSet}$$

In fact, the counit $\omega - G \rightarrow \text{id}$

is an isomorphism, so the embedding

$-G$ is fully faithful.

- Opposite Mackey functor:

$$B_G \longrightarrow B_G^{\text{op}} \xrightarrow{M} \text{Ab}$$

(switching the legs)

Permutation
Mackey functor

"Tambara
functor"
Neil Strickland

By Day Convolution, we endow Mack_G w/ a ^{closed} symmetric monoidal structure:
the left Kan extension

$$\begin{array}{ccc} B_G^{\text{op}} \times B_G^{\text{op}} & \xrightarrow{x} & \text{Ab} \times \text{Ab} \xrightarrow{\otimes_{\mathbb{Z}}} \text{Ab} \\ & \searrow x & \nearrow \square \\ & B_G^{\text{op}} & \end{array}$$

with unit A_G , the Burnside Mackey functor.

DEF (Green functor) commutative monoid obj in $(\text{Mack}_G, \square, A_G)$

DEF (Module over a Green functor R) Mackey functor M with associative, unital $R \times M \rightarrow M$.

Prop A Mackey functor is Green if

- ① $M(x)$ is a commutative ring for $x \in \mathcal{U}_G$
- ② $\text{Res}_H^K: M(G|K) \rightarrow M(G|H)$ makes $M(G|H)$ a $M(G|K)$ -module.
- ③ $\text{Tr}_H^K: M(G|H) \rightarrow M(G|K)$ is a map of $M(G|K)$ -module, i.e.

$$\text{Tr}_H^K(x) \cdot y = \text{Tr}_H^K(x \cdot \text{Res}_H^K(y))$$

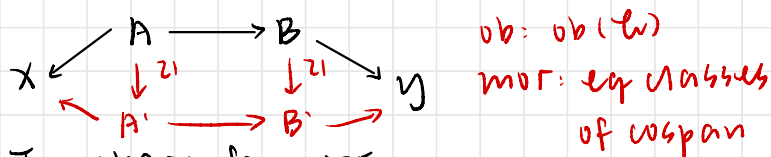
(Frobenius / push-pull relation)

Giving more structures to a Mackey functor

① multiplicative data on the target

DEF (Tambara functor / TNR functor)

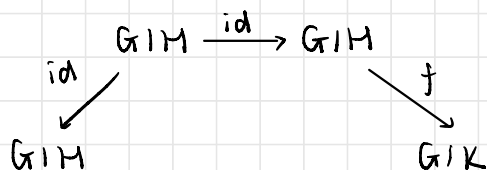
Analogous to $\text{Span}(\mathcal{C})$, define $\text{Cospan}(\mathcal{C})$ w/ morphism



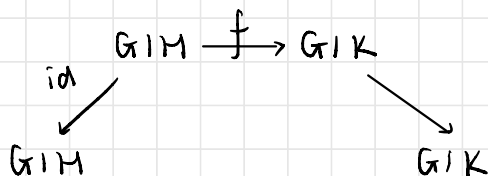
Tambara functor:

$$\text{Cospan}^*(\mathcal{G}\text{SetFin})^{\text{op}} \rightarrow \text{Ab}.$$

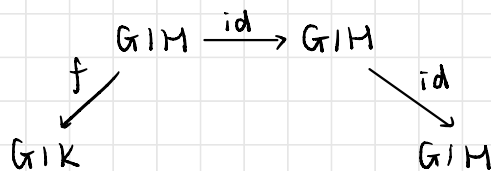
Given $G|I|H, G|I|K \in \mathcal{V}_G$, $f: G|I|H \rightarrow G|I|K$



$$R: \mathcal{V}_G \rightarrow \text{Cospan}(\mathcal{V}_G)^{\text{op}} \rightarrow \text{Ab}$$



$$N: \mathcal{V}_G \rightarrow \text{Cospan}(\mathcal{V}_G)^{\text{op}} \rightarrow \text{Ab}$$



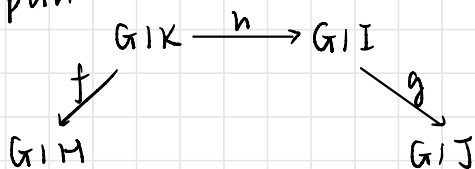
$$T: \mathcal{V}_G^{\text{op}} \rightarrow \text{Cospan}(\mathcal{V}_G)^{\text{op}} \rightarrow \text{Ab}$$

② multiplicative data on the source

Observations $G \mid M \xrightarrow{f} G \mid K \xrightarrow{g} G \mid J$

① $N(fg) = N(f)N(g)$

② cospans



$= T(g)N(h)R(f)$

③ Beck-Chevalley condition:

$$\begin{array}{ccc}
 x' & \xrightarrow{f'} & x \\
 g' \downarrow & \lrcorner & \downarrow g \\
 y' & \xrightarrow{f} & y
 \end{array}
 \quad N(g)T(f') = T(f)N(g')$$

Therefore, a Tambara functor can also be defined as three functors

$T, N: \mathcal{V}_G \rightarrow \mathcal{A}\mathcal{B}$

$R: \mathcal{V}_G^{\text{op}} \rightarrow \mathcal{A}\mathcal{B}$

that satisfy the properties above.

Ex For ~~semiring~~ $M^2 G$, define $\underline{T}_G$

$\underline{T}_G(G \mid M) = M^M$

$\underline{T}_G(x \leftarrow^p A \xrightarrow{r} B \xrightarrow{q} y): \underline{T}(x) \rightarrow \underline{T}(y)$

$\hookrightarrow \mapsto \mapsto y \mapsto \sum_{q(b)=y, r(a)=b} \prod \hookrightarrow p(a)$

$\text{Tam}_G \xrightleftharpoons[\underline{T}_G]{\omega} \text{semiring } G$

(fully faithful embedding)

$RO(G)$ -graded cohomology

DEF (Representation ring)

Grothendieck completion of
finite-dimensional G -rep

$$[V] \oplus [W] = [V \oplus W]$$

$$[V] \cdot [W] = [V \otimes W]$$

Fix a G -universe U . define $RO(G; U)$

obj: finite dimensional G -rep V that
embeds G -equivariantly into U

morph: $\text{Hom}_{RO(G; U)}(V, W)$

$= \{ G\text{-equivariant isometric
isomorphism} \}$ = 1-pt cpr of V

f, g homotopic if $\bar{f}, \bar{g}: S^r \rightarrow S^w$

homotopic.

Define Σ^W :

$$\text{Ho}(RO(G; U)) \times \text{Ho}(G\text{Top}) \rightarrow$$

$$\text{Ho}(RO(G; U)) \times \text{Ho}(G\text{Top})$$

$$(V, X) \mapsto (V \oplus W, S^W \wedge X)$$

DEF ($RO(G)$ -graded cohomology)

$$E: \text{Ho}(RO(G; U)) \times \text{Ho}(G\text{Top})^{\text{op}} \rightarrow \text{Ab}$$

with suspension isomorphism

$$\sigma^W: E^V(X) \rightarrow E^{V \oplus W}(\Sigma^W X)$$

such that

$$① E^v(X \vee Y) = E^v(X) \times E^v(Y)$$

② Exact on cofiber sequence.

$$③ E^v(X) \xrightarrow{\sigma^W} E^{v \oplus W}(\Sigma^W X)$$

$$\begin{array}{ccc} \downarrow \sigma^{W'} & \swarrow & \downarrow \alpha \\ E^{v \oplus W'}(\Sigma^{W'} X) & \xrightarrow{\alpha^{-1}} & E^{v \oplus W'}(\Sigma^W X) \end{array}$$

given isometric isomorphism

$$\alpha: W \rightarrow W'$$

$$④ \sigma_0 = \text{id}, \quad \sigma_{v \oplus W} = \sigma_v \circ \sigma_W$$

EX (Mackey functor valued cohomology)

Fix $X \in \mathcal{J}^G$, $v \in RO(G)$, define $E^v(X)$:

$$\bullet E^v(X)(G/H) = E^v(G/H \times X)$$

• For $K \subseteq H$, covering map

$$\pi: G/K \times X \longrightarrow G/H \times X$$

R_K^H : restriction along π

T_K^H : gysin homomorphism

EX (Equivariant Eilenberg-MacLane sp)

Given a Mackey functor $M \in \text{Mack } G$

define $HM \in \text{Sp}^G$ with

$$\pi_k^H(HM) = \begin{cases} 0, & k \neq 0 \\ M(G/H), & k = 0 \end{cases} \quad \text{RB}$$

The $RO(G)$ -cohomology given by M :

$$H^v(X; M) = [X, \Sigma^v HM]_G$$

Ex For $E \in \text{Sp } G$, let $E_G^V(X) = [X, S^V \wedge E]_G$

This is an $RO(G)$ -graded cohomology theory.

Coefficient system

$$\left\{ \begin{array}{c} \mathcal{V}_G^{ob} \rightarrow \text{Ab} \end{array} \right\}$$

Bredon cohomology

Mack_G

$$\left\{ \begin{array}{c} \text{Span}^+(\mathcal{V}_G)^{op} \\ \downarrow \\ \text{Ab} \end{array} \right\}$$

$RO(G)$ -graded cohomology

$$\begin{array}{ccc} \mathcal{V}_G^{ob} & \xrightarrow{\quad} & \text{Ab} \\ \downarrow & \nearrow ? & \\ \text{Span}^+(\mathcal{V}_G) & & \end{array}$$

TMM (May - Waner)

i.e. Bredon

The ordinary $\mathbb{Z}$ -graded cohomology can be extended to an $RO(G)$ -graded cohomology theory iff its coefficient system extends to a Mackey functor.

Brown's Representability Thm

DEF (compactness)

An object x in a stable category $\mathcal{C}$ is compact if

$$\mathrm{Hom}_{\mathcal{C}}(x, \coprod_i y_i) = \coprod_i \mathrm{Hom}_{\mathcal{C}}(x, y_i)$$

(generating set)

A generating set in a triangulated category $\mathcal{C}$ is closed under shift and consists of objects that detect 0, i.e.

$$x = 0 \text{ iff } \mathrm{Hom}_{\mathcal{C}}(z, x) = 0 \quad \forall z \in \mathcal{C}.$$

EX Dualizable $(G \rightarrow \text{spectra in } \mathrm{Mod}(\mathrm{Sp}^{(G)}))$
are a compact generating set.

THM For a compactly generated

(Brown triangulated category $\mathcal{C}$

Neeman)
 $H: \mathcal{C}^{\mathrm{ob}} \rightarrow \mathrm{Ab}$ w/

$$\textcircled{1} H(\coprod_i x_i) = \prod_i H(x_i)$$

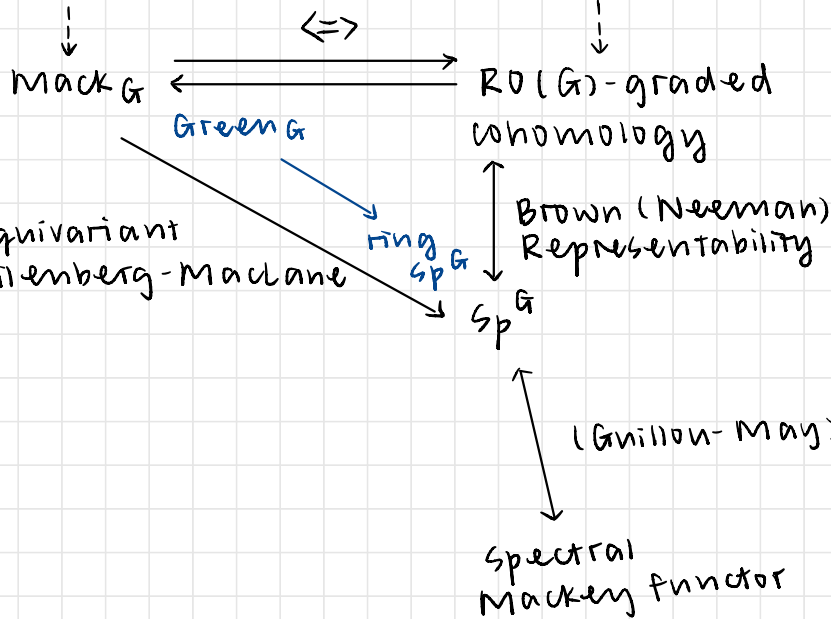
$$\textcircled{2} x \rightarrow y \rightarrow z \rightarrow \Sigma x \text{ is sent to a long exact sequence}$$

Then H is representable.

COR $\mathrm{RO}(G)$ -graded cohomology theories

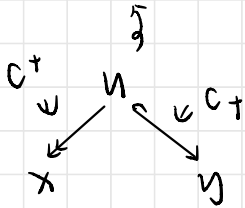
$$\begin{array}{c} \mathcal{Z} \\ \mathcal{Z} \\ \mathcal{Z} \\ \mathrm{Sp}^G \end{array}$$

Coefficient system $\rightarrow$ Bredon cohomology



Rmk: Spectral Mackey functor

$$A^{\text{eff}}(C, C_+, C_+^{\text{op}}) \xrightarrow{\text{op}} Sp$$



∞ -categorical version of Mackey funct

Reference:

Peter Webb: A Guide to Mackey Functor

Neil Strickland: Tambara Functor

Clark Barwick: Spectral Mackey Functors and Equivariant K-theory

Andrew Blumberg: Equivariant Stable Homotopy Theory