Equivariant algebra & RO(G)-graded cohomology

Equivariant algebra

DEF (Span) Given a category $\mathcal{C}$,
$\mathrm{Span}(\mathcal{C})$: ob: $\mathrm{ob}(\mathcal{C})$

mor: $\mathrm{Hom}_{\mathrm{Span}(\mathcal{C})}(X, Y)$

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \Big/ \sim$$

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \sim \begin{tikzcd} & A' \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \quad \text{iff}$$

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[dd, "\simeq"] & \\ X & & Y \\ & A' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Composition

$$\begin{tikzcd} & & A \oplus A' \arrow[dl] \arrow[dr] & & \\ & A \arrow[dl] \arrow[dr] & & A' \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

Disjoint union

$$\begin{tikzcd} & A \amalg A' \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

DEF (Burnside category)

$$B_G := \mathrm{Span}^+(G\mathrm{set}^{\mathrm{fin}})$$

DEF (Mackey functor 1) A Mackey functor is a contravariant functor

$$B_G{}^{op} \longrightarrow Ab$$

Given $G/M, G/K \in \mathcal{O}_G$. $f: G/M \longrightarrow G/K$

$$\begin{tikzcd} & G/M \arrow[dl, "id"'] \arrow[dr, "f"] & \\ G/M & & G/K \end{tikzcd} \qquad \begin{tikzcd} & G/M \arrow[dl, "f"'] \arrow[dr, "id"] & \\ G/K & & G/M \end{tikzcd}$$

$T: \mathcal{O}_G{}^{op} \longrightarrow B_G{}^{op} \longrightarrow Ab \qquad R: \mathcal{O}_G \longrightarrow B_G{}^{op} \longrightarrow Ab$

<u>Observations</u> $\quad G/M \xrightarrow{f} G/K \xrightarrow{g} G/J$

① $T(fg) = T(f)T(g)$

② $R(fg) = R(g)R(f)$

③ Span

$$\begin{tikzcd} & G/K \arrow[dl, "f"'] \arrow[dr, "g"] & \\ G/M & & G/J \end{tikzcd}$$

$$= T(g)R(f) = \begin{tikzcd} & & G/K \arrow[dl] \arrow[dr] & & \\ & G/K \arrow[dl, "f"'] \arrow[dr] & & G/K \arrow[dl] \arrow[dr, "g"] & \\ G/M & & G/K & & G/J \end{tikzcd}$$

④ Beck-Chevalley condition:

$$\begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & x \arrow[d, "g"] \\ y' \arrow[r, "f"] \arrow[ur, dashed] & y \end{tikzcd} \qquad R(g)T(f) = T(f')R(g')$$

Motivates an alternative def of Mackey functor w/o the use of span

DEF (Mackey functor 2)

$$\begin{cases} M_*: \mathcal{V}_G \longrightarrow Ab \\ M^*: \mathcal{V}_G^{op} \longrightarrow Ab \end{cases}$$

① $M_*(X) = M^*(X)$

② Given

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "g"] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

$M^*(g)M_*(f)$
$= M_*(f')M^*(g')$

③ For $X \longrightarrow X \amalg Y \longleftarrow Y$

$$M_*(X) \oplus M_*(Y) = M_*(X \amalg Y)$$

DEF (Mackey functor 3)

A function $M : \vartheta_G \longrightarrow Ab$ with

$$\begin{cases} R^H_K : M(H) \to M(K) \\ T^H_K : M(K) \longrightarrow M(H) \\ c_g : M(H) \longrightarrow M(gHg^{-1}),\ g \in G \end{cases}$$

G abelian
$K \subseteq H$
$G/K \to G/H$

① $R^H_H,\ T^H_H,\ c_h : M(H) \to M(H)$

are identity for $h \in H$.

② $R^J_H = R^K_H R^J_K,\quad T^J_H = T^J_K T^K_H,\quad H \subseteq K \subseteq J$

③ $c_{gg'} = c_g c_{g'}$.

④ $c_g R^K_H = R^{gKg^{-1}}_{gHg^{-1}} c_g,\quad c_g T^K_H = T^{gKg^{-1}}_{gHg^{-1}} c_g.$

⑤ $R^J_K T^J_H$

$$= \bigoplus_{x \in K\backslash J/H} T^K_{K \cap xHx^{-1}}\, c_x\, R^H_{H \cap x^{-1}Kx}$$

$H, K \subseteq J$

Axiomatic def of Mackey functor

not abelian

$K \subseteq xHx^{-1}$

$$\coprod_{x \in [H\backslash J/K]} J/(H \cap xKx^{-1})$$

$$\begin{tikzcd} J/H \times J/K \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & J/H \arrow[d] \\ J/K \arrow[r] & J/J \end{tikzcd}$$

EX (Mackey functor)

- constant $\mathbb{Z}$: R^H_K id $\quad T^H_K$ $\times |H|K|$

$K \subseteq xHx^{-1}$

- $\pi_n^{-}(X) : G/H \longrightarrow \pi_n^H(X)$

- Burnside Mackey functor A_G

$A_G(G/H) = \{H\text{Set}^{fin}, \amalg\}^+$

R^H_K : restriction along $K \longrightarrow H$

$T^H_K : H \times_K -$

- Representation R_G

$R_G(G/H) = \{\text{fin-dim } H\text{-rep}\}^+$

R^H_K: restriction along $K \longrightarrow H$

$T^H_K : \mathbb{Z}[H] \times_{\mathbb{Z}[K]} -$

- Cohomological Mackey functor:

if $T^H_K R^H_K : M(H) \to M(H)$ is $\times |H/K|$

  - constant
  - group cohomology $H^n(G, M)$
  - Tate cohomology $\hat{H}^n(G, M)$
  - $G = Gal(L/K)$, $H \subseteq G$

    Let $M_G(G/H) = \mathcal{O}_{L^H}$

- For a set $S \supseteq G$

$\underline{M}_G(G/H) = Map^G(G/H, \mathbb{Z}[S]) = \mathbb{Z}[S]^H$

$\underline{M}_G(X \xleftarrow{p} A \xrightarrow{q} Y) : \underline{M}(X) \to \underline{M}(Y)$

$$s \mapsto \left(y \mapsto \sum_{q(a)=y} s(p(a))\right)$$

Permutation Mackey functor

$$\begin{tikzcd} Mack_G \arrow[r, shift left, "w"] \arrow[r, phantom, "\perp"] & GSet \arrow[l, shift left, "\underline{M}_G"] \end{tikzcd}$$

In fact, the counit $w\underline{\ \ }_G \to id$ is an isomorphism, so the embedding $\underline{\ \ }_G$ is fully faithful.

"Tambara functor"

Neil Strickland

- Opposite Mackey functor:

$B_G \longrightarrow B_G{}^{op} \xrightarrow{M} Ab$

(switching the legs)

By Day Convolution, we endow $Mack_G$ w/ a closed symmetric monoidal structure: the left Kan extension

$$\begin{tikzcd} BG^{op} \times BG^{op} \arrow[r, "\times"] \arrow[dr, "\times"'] & Ab \times Ab \arrow[r, "\otimes_{\mathbb{Z}}"] & Ab \\ & BG^{op} \arrow[ur, "\Box"'] & \end{tikzcd}$$

with unit A_G, the Burnside Mackey functor.

Giving more structures to a Mackey functor

<u>DEF</u> (Green functor) commutative monoid obj in $(Mack_G, \Box, A_G)$

① multiplicative data on the target

<u>DEF</u> (Module over a Green functor R) Mackey functor M with associative, unital $R \times M \longrightarrow M$.

<u>Prop</u> A Mackey functor is Green if

① $M(x)$ is a commutative ring for $x \in \mathcal{O}_G$

② $Res^K_H : M(G/K) \longrightarrow M(G/H)$ makes $M(G/H)$ a $M(G/K)$-module.

③ $Tr^K_H : M(G/H) \longrightarrow M(G/K)$ is a map of $M(G/K)$-module, i.e.

$$Tr^K_H(x) \cdot y = Tr^K_H(x \cdot Res^K_H(y))$$

(Frobenius / push-pull relation)

<u>DEF</u> (Tambara functor / TNR functor)

Analogous to Span($\mathcal{C}$), define Cospan($\mathcal{C}$) w/ morphism

$$\begin{tikzcd}
 & A \arrow[r] \arrow[dl] \arrow[d, "\cong"] & B \arrow[dr] \arrow[d, "\cong"] & \\
X & A' \arrow[l] \arrow[r] & B' \arrow[r] & Y
\end{tikzcd}$$

ob: ob($\mathcal{C}$)
mor: eq classes of cospan

Tambara functor:

$$\mathrm{Cospan}^{+}(G\mathrm{set}\ \mathrm{fin})^{op} \longrightarrow Ab.$$

② multiplicative data on the source

Given G/H, $G/K \in \mathcal{O}_G$. $f: G/H \longrightarrow G/K$

$$\begin{tikzcd}
 & G/H \arrow[r, "id"] \arrow[dl, "id"'] & G/H \arrow[dr, "f"] & \\
G/H & & & G/K
\end{tikzcd}$$

$$R: \mathcal{O}_G \longrightarrow \mathrm{Cospan}(\mathcal{O}_G)^{op} \longrightarrow Ab$$

$$\begin{tikzcd}
 & G/H \arrow[r, "f"] \arrow[dl, "id"'] & G/K \arrow[dr] & \\
G/H & & & G/K
\end{tikzcd}$$

$$N: \mathcal{O}_G \longrightarrow \mathrm{Cospan}(\mathcal{O}_G)^{op} \longrightarrow Ab$$

$$\begin{tikzcd}
 & G/H \arrow[r, "id"] \arrow[dl, "f"'] & G/H \arrow[dr, "id"] & \\
G/K & & & G/H
\end{tikzcd}$$

$$T: \mathcal{O}_G^{op} \longrightarrow \mathrm{Cospan}(\mathcal{O}_G)^{op} \longrightarrow Ab$$

Observations $\quad G/M \xrightarrow{f} G/K \xrightarrow{g} G/J$

1. $N(fg) = N(f)N(g)$
2. cospan

$$\begin{tikzcd} & G/K \arrow[r, "h"] \arrow[dl, "f"'] & G/I \arrow[dr, "g"] & \\ G/M & & & G/J \end{tikzcd}$$

$= T(g)N(h)R(g)$

3. Beck-Chevalley condition:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\ Y' \arrow[r, "f"] & Y \end{tikzcd} \qquad N(g)T(f) = T(f')N(g')$$

Therefore, a Tambara functor can also be defined as three functors

$T, N : \mathcal{O}_G \to Ab$

$R : \mathcal{O}_G^{op} \to Ab$

that satisfy the properties above.

Ex For ~~semi~~ring $M \curvearrowleft G$, define $\underline{I}_G$

$\underline{I}_G(G/H) = M^H$

$\underline{I}_G(X \xleftarrow{p} A \xrightarrow{r} B \xrightarrow{q} Y) : \underline{I}(X) \to \underline{I}(Y)$

$$s \mapsto \Big(y \mapsto \sum_{q(b)=y} \prod_{r(a)=b} s(p(a))\Big)$$

$$\begin{tikzcd} \mathrm{Tam}_G \arrow[r, shift left, "\omega"] \arrow[r, phantom, "\perp"] & \cancel{\text{semi}}\text{ring}_G \arrow[l, shift left, "\underline{I}_G"] \end{tikzcd}$$

(fully faithful embedding)

RO(G)-graded cohomology

DEF (Representation ring)
Grothendieck completion of
finite-dimensional G-rep

$[V] \oplus [W] = [V \oplus W]$

$[V] \cdot [W] = [V \otimes W]$

Fix a G-universe $\mathcal{U}$. define $RO(G;\mathcal{U})$

obj: finite dimensional G-rep V that
embeds G-equivariantly into $\mathcal{U}$

morph: $\mathrm{Hom}_{RO(G;\mathcal{U})}(V, W)$
$:= \{$G-equivariant isometric isomorphism$\}$

f, g homotopic if $\bar{f}, \bar{g} : S^V \to S^W$ ($= 1$-pt cpt of V)
homotopic.

Define Σ^W:
$$Ho(RO(G;\mathcal{U})) \times Ho(GTop) \to Ho(RO(G;\mathcal{U})) \times Ho(GTop)$$
$$(V, X) \mapsto (V \oplus W, S^W \wedge X)$$

DEF (RO(G)-graded cohomology)

$$E : Ho(RO(G;\mathcal{U})) \times Ho(GTop)^{op} \to Ab$$

with suspension isomorphism
$$\sigma^W : E^V(X) \to E^{V \oplus W}(\Sigma^W X)$$

such that

① $E^V(X \vee Y) = E^V(X) \times E^V(Y)$

② Exact on cofiber sequence.

③
$$\begin{tikzcd}
E^V(X) \arrow[r, "\sigma_W"] \arrow[d, "\sigma_{W'}"'] \arrow[dr, phantom, "\circlearrowleft"] & E^{V\oplus W}(\Sigma^W X) \arrow[d, "\alpha"] \\
E^{V\oplus W'}(\Sigma^{W'} X) \arrow[r, "\alpha^{-1}"] & E^{V\oplus W'}(\Sigma^W X)
\end{tikzcd}$$

given isometric isomorphism
$\alpha : W \to W'$.

④ $\sigma_0 = \mathrm{id}$, $\sigma_{V\oplus W} = \sigma_V \circ \sigma_W$

EX (Mackey functor valued cohomology)

Fix $X \in \mathcal{S}^G$, $V \in RO(G)$, define $\underline{E}^V(X)$:

- $\underline{E}^V(X)(G/H) = E^V(G/H \times X)$
- For $K \subseteq H$, covering map
$$\pi : G/K \times X \longrightarrow G/H \times X$$
R^H_K: restriction along π
T^H_K: gysin homomorphism

EX (Equivariant Eilenberg-MacLane sp)

Given a Mackey functor $M \in \mathrm{Mack}_G$
define $HM \in Sp^G$ with
$$\pi^H_k(HM) = \begin{cases} 0, & k \neq 0 \\ M(G/H) \in \mathrm{Ab}, & k = 0 \end{cases}$$
The $RO(G)$-cohomology given by M:
$H^V(X; M) = [X, \Sigma^V HM]_G$

__Ex__ For $E \in Sp^G$, let $E^V_G(X) = [X, S^V \wedge E]_G$

This is an $RO(G)$-graded cohomology theory.

$$\begin{array}{ccc} \text{Coefficient system} & & \text{Mack}_G \\ \rightsquigarrow \quad \mathcal{O}_G^{ob} \to Ab & & \rightsquigarrow \quad \begin{array}{c} \text{Span}^+(\mathcal{O}_G)^{op} \\ \downarrow \\ Ab \end{array} \\ \text{Bredon cohomology} & & RO(G)\text{-graded cohomology} \end{array}$$

$$\begin{tikzcd} \mathcal{O}_G^{ob} \arrow[r] \arrow[d] & Ab \\ \text{Span}^+(\mathcal{O}_G) \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

__Thm__ (May-Waner)

The ordinary $\mathbb{Z}$-graded cohomology (i.e. Bredon) can be extended to an $RO(G)$-graded cohomology theory iff its coefficient system extends to a Mackey functor.

Brown's Representability thm

DEF (compactness)

An object x in a stable category $\mathcal{C}$ is compact if

$$\mathrm{Hom}_{\mathcal{C}}(x, \coprod_i y_i) = \coprod_i \mathrm{Hom}_{\mathcal{C}}(x, y_i)$$

(generating set)

A generating set in a triangulated category $\mathcal{C}$ is closed under shift and consists of objects that detect 0, i.e. $x = 0$ iff $\mathrm{Hom}_{\mathcal{C}}(z, x) = 0 \ \forall z \in \mathcal{C}$.

EX Dualizable (G-) spectra in $\mathrm{Ho}(Sp^{(G)})$ are a compact generating set.

THM (Brown Neeman) For a compactly generated triangulated category $\mathcal{C}$

$H : \mathcal{C}^{op} \to \mathrm{Ab}$ w/

① $H(\coprod_i X_i) = \prod_i H(X_i)$

② $X \to Y \to Z \to \Sigma X$ is sent to a long exact sequence

Then H is representable.

Cor $RO(G)$-graded cohomology theories

$$\Big\updownarrow \wr$$

Sp^G

$$\begin{tikzcd}
\text{Coefficient system} \arrow[r] \arrow[d, dashed] & \text{Bredon cohomology} \arrow[d, dashed] \\
\mathrm{Mack}_G \arrow[r, shift left, ""{name=U}] \arrow[dr, "\text{Equivariant Eilenberg-MacLane}"'] & \text{RO(G)-graded cohomology} \arrow[l, shift left, ""{name=D}] \arrow[d, leftrightarrow, "\text{Brown (Neeman) Representability}"] \\
\mathrm{Green}_G \arrow[r, "\text{ring } Sp^G"'] & Sp^G \arrow[d, leftrightarrow, "\text{(Guillon-May)}"] \\
& \text{Spectral Mackey functor}
\arrow[from=U, to=D, phantom, "\Longleftrightarrow"]
\end{tikzcd}$$

Rmk: spectral Mackey functor

$$A^{eff}(C, C_\dagger, C^\dagger)^{op} \longrightarrow Sp$$

$$\rotatebox{90}{$\in$}$$

$$\begin{tikzcd}
& U \arrow[dl, "C^\dagger" description] \arrow[dr, hook, "C_\dagger"] & \\
X & & Y
\end{tikzcd}$$

∞-categorical version of Mackey funct

Reference:

Peter Webb: A Guide to Mackey Functor

Neil Strickland: Tambara Functor

Clark Barwick: Spectral Mackey Functors and Equivariant K-theory

Andrew Blumberg: Equivariant Stable Homotopy Theory