A guide to the literature on topological K-theory

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The purpose of this document is to ease the process of finding where certain facts are proven or what certain references contain within the topological K-theory literature. Alternatively, it can also be used to accelerate the process of surveying and learning specific results or broad topics in topological K-theory. This document was started (circa 2023) as a way to help keep track of references that give particular attention to the real theoreis, KO or KR, but it has since expanded to include anything related to topological K-theory, with specific sections devoted to KO-theory and KR-theory. Some references are included whose principal topic is not K-theory; in such cases, we typically include comments on its relevance to K-theory.

This document is a work in progress, and, at any given point in time, reflects the mathematical biases of the author. Likewise, there are many relevant and important references that have not yet been included; this will improve over time (see Section 15 for some that are planned). Suggestions that can help improve this document's usefulness and reliability are very welcome and appreciated, and can be sent to ykamel2@illinois.edu.

Organization and conventions. The document opens in Section 1 with a roughly chronological list of all cited references that are in some way explicitly related to topological K-theory. Section 2 is home to all remaining cited references, which are included due to their importance in the development of topological K-theory, despite not dealing with it specifically. Sections 1 and 2 are separated so that Section 1 can serve as a kind of history of topological K-theory. Following Section 2, all cited references are sorted into subtopics of topological K-theory, for ease of surveying a particular subset of the literature. Finally, Section 15 contains references that have not yet been incorporated into the main part of the document (or are missing important information or details), but with future plans to do so. The document can be navigated using a variety of **green links**: the table of contents, the citations, and the "cited-on" page numbers in the bibliography (which, thanks to a tip from Howard Beck, always link back to an entry that discusses the reference linked). In each section, references are sorted by year, and within each year, by authors' last names. In this document, "K-theory" is a generic term that refers to the subject dealing with KU, KO, and KR (and sometimes other versions of K-theory, when explicitly stated). In particular, "K-theory" is not synonymous with "KU-theory" or "algebraic K-theory", in contrast to common conventions.

Contents

1 Topological K-theory by date

This section contains all references cited in this document that contain content explicitly about topological K-theory. Accordingly, it can be seen as a “history” of the subject.

★ [56] 1957. Bott, *The stable homotopy of the classical groups.*
Proves Bott periodicity for the first time, using Morse theory.

★ [27] 1959. Atiyah–Hirzebruch, *Riemann-Roch theorems for differentiable manifolds.*
Defines topological K-theory for the first time.

★ [57] 1959. Bott, *Quelques remarques sur les théorèmes de périodicité.*
Proves “strong” Bott periodicity, i.e. that the periodicity in KU- or KO-cohomology is induced by multiplication by a particular element. Note that the standard Bott periodicity theorem about the homotopy types of $\Omega^k\,\mathrm{U}$ and $\Omega^k\,\mathrm{O}$ don’t depend on any multiplicative structure of the associated spectra, KU and KO. It turns out that the periodicity theorem is, in fact, related to the natural multiplicative structures that the K-theory spectra carry.

★ [22] 1961. Atiyah, *Characters and cohomology of finite groups.*
Proves a predecessor to the Atiyah–Segal completion theorem [33] , namely the case where $X = *$ and G is a finite group. Also relates this result to the group cohomology of G.

★ [28] 1961. Atiyah–Hirzebruch, *Vector bundles and homogeneous spaces.*
Develops the basic properties of K-theory as a cohomology theory. Introduces the Atiyah–Hirzebruch spectral sequence. Written in terms of KU, but many constructions carry over to KO. Shows that the Chern character induces an isomorphism,

$$\mathrm{KU}^*(X) \otimes \mathbb{Q} \cong \mathrm{H}^*(X;\mathbb{Q}),$$

where the $*$ represents a $\mathbb{Z}/2$-grading on both sides (on the left side, this is induced by Bott periodicity, and on the right side, it is induced by the $\mathbb{Z}$-grading via the quotient $\mathbb{Z} \to \mathbb{Z}/2$). In particular, $\mathrm{KU}^0(X) \otimes \mathbb{Q} \cong \mathrm{H}^{\mathrm{even}}(X;\mathbb{Q})$ and $\mathrm{KU}^1(X) \otimes \mathbb{Q} \cong \mathrm{H}^{\mathrm{odd}}(X;\mathbb{Q})$. Also proves a special case of the Atiyah–Segal completion theorem [33] , namely when $X = *$ and G is a compact connected Lie group, relating the representation ring of G to $\mathrm{KU}^*(\mathrm{B}G)$.
See [174] for a refinement of the relation between K-theory and ordinary cohomology that includes torsion and an application to KO-theory. See [77] for results on generalizations of the Chern character to arbitrary cohomology theories.

★ [1] 1962. Adams, *Vector fields on spheres.*
Introduces certain power operations in KU-theory and KO-theory (now called “Adams operations”) and uses them to determine the maximum number of linearly independent vector fields on each sphere S^n. Notably, the computations involving KO-theory are essential to the

proof; consideration of KU-theory alone is insufficient. Also computes stuff about projective spaces. See [24] for a more thorough analysis of power operations in K-theory from the perspective of equivariant K-theory. write some details out here.

★ [145] 1963. Milnor, *Morse theory.*
Contains proofs of both complex and real Bott periodicity by identifying spaces of minimal geodesics in orthogonal groups with spaces of Clifford module structures. The Clifford language is not used explicitly; instead, the language of "commuting complex structures" is used, where a complex structure is the same as an action by a Clifford algebra generator.

★ [7] 1964. Anderson, *The real K-theory of classifying spaces.*
Computes $\mathrm{KO}^*(\mathrm{B}G)$ for a compact connected Lie group, G.

$$\mathrm{KO}^i(\mathrm{B}G) \cong \begin{cases} \mathrm{RO}(G)^\wedge, & i=0 \\ 0, & i=1 \\ \mathrm{RU}(G)^\wedge/\,\mathrm{RO}(G)^\wedge, & i=2 \\ \mathrm{RSp}(G)^\wedge/\,\mathrm{RU}(G)^\wedge, & i=3 \\ \mathrm{RSp}(G)^\wedge \cong \mathrm{KSp}^0(\mathrm{B}G), & i=4 \\ 0, & i=5 \\ \mathrm{RU}(G)^\wedge/\,\mathrm{RSp}(G)^\wedge, & i=6 \\ \mathrm{RO}(G)^\wedge/\,\mathrm{RU}(G)^\wedge, & i=7. \end{cases} \quad \bmod 8$$

These computations also follow from [33] .

★ [40] 1964. Atiyah, *K-theory.*
A book that covers the basics of complex K-theory (KU-theory) without assuming knowledge of algebraic topology. The real case (KO) is not discussed, but it is a very useful text for understanding the ideas of both theories. Includes a proof of complex Bott periodicity, and a proof of the Thom isomorphism in KU-theory for complex vector bundles, inducing a complex orientation $\mathrm{MU} \to \mathrm{KU}$. See [26] and [23] for the Thom isomorphisms for spin, spin^c, and Real vector bundles, and see [25] for the Thom isomorphisms for Real spin^c vector bundles, as well as for equivariant versions of all of the above Thom isomorphisms. Also contains an appendix proving (independently from [114]) the "Atiyah–Jänich theorem", namely, that the index induces an isomorphism,

$$\mathrm{index} : [X, \mathcal{F}] \to \mathrm{KU}^0(X),$$

where $\mathcal{F}$ is the space of Fredholm operators on a separable complex Hilbert space.

★ [26] 1964. Atiyah–Bott–Shapiro, *Clifford modules.*

Develops the basic algebraic features of Clifford algebras and modules that are relevant for topological K-theory, especially for treating the KO and KU cases uniformly. In particular, they establish an isomorphism between the graded ring KO_* (KU_*) and a graded ring of equivalence classes[1] of finite-dimensional virtual real (complex) Clifford modules, under direct sum and tensor product of modules. They realize this construction more generally to construct Thom classes in KO-theory (KU-theory) for vector bundles with a spin (spin^c) structure, which results in a multiplicative map of ring spectra, $\mathrm{MSpin} \to \mathrm{KO}$ ($\mathrm{MSpin}^c \to \mathrm{KU}$). After taking homotopy groups, this map recovers the KO_*-valued (KU_*-valued) $\hat{A}$-genus (Todd genus), $\mathrm{MSpin}_* \to \mathrm{KO}_*$ ($\mathrm{MSpin}^c_* \to \mathrm{KU}_*$).

★ [114] 1965. Jänich, *Vektorraumbündel und der Raum der Fredholm Operatoren.*
Proves (independently from [40]) the "Atiyah–Jänich theorem", namely, that the index induces an isomorphism,

$$\mathrm{index} : [X, \mathcal{F}] \to \mathrm{KU}^0(X),$$

where $\mathcal{F}$ is the space of Fredholm operators on a separable complex Hilbert space.

★ [5] 1966. Adams, *On the groups J(X)–IV.*
Identifies a family of nontrivial elements in the stable homotopy groups of spheres, $\pi_* \mathbb{S}$, using KO-theory.

★ [10] 1966. Anderson–Brown–Peterson, *Spin cobordism.*
An announcement of [9] . Describes the additive structure and some of the multiplicative structure of the spin cobordism ring, MSpin_*, using KO-theory.

★ [8] 1966. Anderson–Brown–Peterson, *SU-cobordism, KO-characteristic numbers, and the Kervaire invariant.*
Introduces KO-valued characteristic classes and uses them to study the SU-cobordism ring, MSU_*. Specifically, for each $i \in \mathbb{Z}_{\geq 0}$, the i-th KO-Pontrjagin class $\pi^i \in \mathrm{KO}^0(\mathrm{BSO}(2m))$ is defined as follows. Let $\mathrm{U}(1)^m \hookrightarrow \mathrm{SO}(2m)$ be the maximal torus, let X_j be the complex line bundle on $\mathrm{BU}(1)^m$ classified by the projection to the j-th factor of $\mathrm{BU}(1)$, and set $x_j = [X_j] - 1 \in \mathrm{KU}^0(\mathrm{BU}(1)^m)$. Define

$$\pi_t = \prod_{j=1}^{m} (1 + t(x_j + \bar{x}_j)) \in \mathrm{KU}^0(\mathrm{BU}(1)^m)[t],$$

where $\bar{x}_j = \frac{-x_j}{1+x_j}$ denotes the complex conjugate, and let $\tilde{\pi}^i$ be the $\mathrm{KU}^0(\mathrm{BU}(1)^m)$-valued coefficient of t^i in π_t. Then $\tilde{\pi}^i$ lifts to a unique element $\pi^i \in \mathrm{KO}^0(\mathrm{BSO}(2m))$ called the *i-th* KO-*Pontrjagin class.* Results include:

[1] a Clifford module is equivalent to zero if its module structure can be extended over a larger Clifford algebra.

- An explicit comparison between the KO-Pontrjagin classes and the ordinary Pontrjagin classes of a vector bundle via the Pontrjagin (a.k.a. Chern) character map $\mathrm{KO}^0(X) \to \mathrm{H}^*(X;\mathbb{Q})$.
- Identifies when a class $[M] \in \mathrm{MSU}_*$ is zero in terms of the Chern numbers and KO-characteristic numbers of M.
- Identifies which classes in MSU_* contain framed manifolds, in other words, identifies the image of the unit map, $u : \mathbb{S} \to \mathrm{MSU}$, on homotopy groups. More specifically,

$$\mathrm{Im}(u_k) = \begin{cases} \mathbb{Z}/2, & k = 1,2 \\ 0, & k \neq 1,2 \end{cases} \mod 8.$$

- Contains another proof that the Kervaire invariant of an $(8r+2)$-dimensional framed manifold is zero.

The KO-characteristic classes introduced here are also the source of the splitting of MSpin found in [9] .

★ [23] 1966. Atiyah, *K-theory and reality.*
Introduces KR as a C_2-equivariant cohomology theory, sometimes called "Real K-theory", built from complex vector bundles with Real structure, that restricts to KO-theory on the subcategory of C_2-spaces with trivial C_2-action. Shows that as an $\mathrm{RO}(C_2)$-graded cohomology theory, KR is $(1,1)$-periodic (i.e. $\mathbb{C}$-periodic). More generally, shows that KR has Thom isomorphisms for Real vector bundles, i.e. complex vector bundles with Real structure. This means that KR has a Real orientation, i.e. a ring map, $\mathrm{MU}_{\mathbb{R}} \to \mathrm{KR}$, from the Real bordism spectrum introduced in [153] .

★ [24] 1966. Atiyah, *Power operations in K-theory.*
Systematically studies power operations in KU-theory via taking (nth) tensor powers of vector bundles viewed as elements in (Σ_n-) equivariant KU-theory, which relates K-theory operations to the representation rings of symmetric groups. Compares the Adams operations of [1] to the Steenrod operations via the Atiyah–Hirzebruch spectral sequence and the Chern character [28] .

★ [70] 1966. Conner–Floyd, *The relation of cobordism to K-theories.*
Establishes the "Conner–Floyd isomorphisms",

$$\mathrm{MU}^*(X) \otimes_{\mathrm{MU}^*} \mathrm{KU}^* \cong \mathrm{KU}^*(X), \quad \mathrm{MSp}^*(X) \otimes_{\mathrm{MSp}^*} \mathrm{KO}^* \cong \mathrm{KO}^*(X);$$

$$\text{or } \mathrm{MU}_*(X) \otimes_{\mathrm{MU}_*} \mathrm{KU}_* \cong \mathrm{KU}_*(X), \quad \mathrm{MSp}_*(X) \otimes_{\mathrm{MSp}_*} \mathrm{KO}_* \cong \mathrm{KO}_*(X).$$

In other words, "complex cobordism determines complex K-theory" and "symplectic cobordism determines KO-theory".

★ [9] 1967. Anderson–Brown–Peterson, *The structure of the spin cobordism ring.* Contains the results that were announced in [10] . Describes the structure of the spin cobordism ring, MSpin_*, in terms of connective covers of KO-theory and suspensions of $\mathrm{H}\mathbb{Z}/2$. More specifically, let $\pi^i : \mathrm{BSpin} \to \mathrm{KO}$ be the pullback to BSpin of the i-th KO-Pontrjagin class (with the same name) defined in [8] , and let $\pi^I = \pi^{i_1} \dots \pi^{i_k}$, for $I = (i_1, ..., i_k)$. Results include:

- The map $\pi^I : \mathrm{BSpin} \to \mathrm{KO}$ factors as $\pi^I : \mathrm{BSpin} \to \mathrm{ko}\langle n_I \rangle \to \mathrm{KO}$, where
$$n_I = \begin{cases} 4|I|, & |I| \text{ is even} \\ 4|I| - 2, & |I| \text{ is odd,} \end{cases}$$
where $|I| = i_1 + \cdots + i_k$.

- Let $f_I : \mathrm{MSpin} \to \mathrm{ko}\langle n_I \rangle$ be the map that corresponds to π^I under the Thom isomorphism [26] . There exists a collection of elements $Z \subset \mathrm{H}^*(\mathrm{MSpin}; \mathbb{Z}/2)$, such that the map,
$$F_{\mathrm{ABP}} : \mathrm{MSpin} \xrightarrow{\left(\prod f_I\right) \times \left(\prod z\right)} \prod_{\substack{|I| \text{ even} \\ 1 \notin I}} \mathrm{ko}\langle 4|I| \rangle \times \prod_{\substack{|I| \text{ odd} \\ 1 \notin I}} \mathrm{ko}\langle 4|I| - 2 \rangle \times \prod_{z \in Z} \Sigma^{|z|} \mathrm{H}\,\mathbb{Z}/2,$$
induces an isomorphism in cohomology with $\mathbb{Z}/2$-coefficients. Since both the domain and the codomain of F_{ABP} are connective with (degree-wise) finitely generated homotopy groups, F_{ABP} is a 2-local equivalence.

- A(n unoriented) manifold M is (unoriented) cobordant to a spin manifold if and only if all Stiefel–Whitney numbers of M involving w_1 or w_2 vanish.

While the complex case is not discussed in this paper, the same methods and constructions apply to produce a 2-local splitting of MSpin^c (see [172] for details). Namely, let π^I also denote the composite $\mathrm{BSpin}^c \to \mathrm{BSO} \xrightarrow{\pi^I} \mathrm{KO} \xrightarrow{\otimes \mathbb{C}} \mathrm{KU}$.

- The map π^I factors as $\pi^I : \mathrm{BSpin}^c \to \mathrm{ku}\langle 4|I| \rangle \to \mathrm{KU}$.

- Let $f_I : \mathrm{MSpin}^c \to \mathrm{ku}\langle 4|I| \rangle$ be the map that corresponds to π^I under the Thom isomorphism [26] . There exists a collection of elements $Z \subset \mathrm{H}^*(\mathrm{MSpin}^c; \mathbb{Z}/2)$, such that the map,
$$F^c_{\mathrm{ABP}} : \mathrm{MSpin}^c \xrightarrow{\left(\prod f_I\right) \times \left(\prod z\right)} \prod_I \mathrm{ku}\langle 4|I| \rangle \times \prod_{z \in Z} \Sigma^{|z|} \mathrm{H}\,\mathbb{Z}/2,$$
is a 2-local equivalence.

★ [12] 1968. Anderson–Hodgkin, *The K-theory of Eilenberg–MacLane complexes.*
Computes the K-theory of Eilenberg–Mac Lane spaces, $\mathrm{KU}^*(K(\pi, n))$. Also includes an appendix which contains results for $\mathrm{KO}^*(K(\pi, n))$. Write the actual results in both cases.

★ [25] 1968. Atiyah, *Bott periodicity and the index of elliptic operators.*
Uses index theory to prove generalizations of the 8-fold Bott periodicity of KO, the 2-fold Bott periodicity of KU, and the $(1,1)$-fold Bott periodicity of KR in the equivariant settings. The most general statement from which all others follow is the following.

- Let $G \to \mathrm{Spin}^c(p, q)$ be a C_2-equivariant homomorphism of groups with C_2-action, and suppose $p \equiv q \mod 8$. Then there is a class $u \in \mathrm{KR}_G(\mathbb{R}^{p,q})$, such that multiplication by u induces an isomorphism,

$$\mathrm{KR}_G(X) \cong \mathrm{KR}_G(\mathbb{R}^{p,q} \times X).$$

  Here, C_2 acts on $\mathbb{R}^{p,q} = \mathbb{R}^{p+q}$ trivially on the $\mathbb{R}^p$ factor and by -1 on the $\mathbb{R}^q$ factor, and $\mathrm{Spin}^c(p, q)$ is the spin group in the Real Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{R}^{p,q})$ with the induced C_2-action.

The corresponding Thom isomorphisms are formal consequences of the equivariant periodicity theorems.

★ [35] 1968. Atiyah–Singer, *The index of elliptic operators: I.*
Part I of a five part series on the "Atiyah–Singer index theorem". Reviews Thom classes in KU-theory and KU_G-theory via the exterior algebra construction (see [40] and [165]), along with the induced topological index

$$\text{t-ind} : \mathrm{KU}^0_G(TX) \to \mathrm{KU}^0_G(*) \cong \mathrm{RU}(G),$$

where X is a compact smooth G-manifold. Defines the notion of an *index function* to be a natural transformation $\mathrm{KU}^0_{(-)}(T(-)) \to \mathrm{KU}^0_{(-)}(*) \cong \mathrm{RU}(-)$ of functors between appropriately defined categories that satisfies certain axioms. Shows that there is exactly one index function, equal to t-ind. Reviews facts about psuedo-differential operators and elliptic operators over X, and defines the symbol of an elliptic operator as an element of $\mathrm{KU}^0_G(TX)$. Defines the index of a G-invariant[2] elliptic operator P over X as

$$[\mathrm{Ker}(P)] - [\mathrm{Coker}(P)] \in \mathrm{KU}^0_G(*) \cong \mathrm{RU}(G),$$

and explains how the index of P only depends on its symbol in $\mathrm{KU}_G(TX)$. Thus, the index defines a map

$$\text{a-ind} : \mathrm{KU}^0_G(TX) \to \mathrm{KU}^0_G(*) \cong \mathrm{RU}(G),$$

[2] elliptic operators which are not G-invariant can be replaced with a G-invariant one with the same symbol.

called the analytic index. Shows that a-ind is an index function, thus showing that it equals t-ind,

$$\text{a-ind} = \text{t-ind} : \mathrm{KU}^0_G(TX) \to \mathrm{KU}^0_G(*).$$

As a consequence, a topological formula is obtained for the index of an elliptic operator on a G-manifold in terms of topological K-theory.

★ [118] 1968. Karoubi, *Algèbres de Clifford et K-théorie.*
Explores further the relationship between Clifford algebras and K-theory introduced in [26] . Includes both the real and complex cases. need to add more details here.

★ [165] 1968. Segal, *Equivariant K-theory.*
An introduction to equivariant K-theory built from G-equivariant vector bundles. This work is Segal's thesis, and much of it is attributed to Atiyah.

★ [172] 1968. Stong, *Notes on cobordism theory.*
An exposition of many topics in cobordism theory, including detailed treatments of various relationships between spin and spin^c bordism with topological K-theory (Chapter XI). The following link contains a detailed table of contents which was compiled and made available by Landweber and Ravenel. Contents include:

- a computation of $\mathrm{H}^*(\mathrm{BSpin}; \mathbb{Z}/2)$ (p. 291-292) and $\mathrm{H}^*(\mathrm{BSpin}^c; \mathbb{Z}/2)$ (p. 293);
- if a vector bundle admits a KO-orientation, then it must admit a spin structure (p. 301);
- an exposition of the main results of Anderson–Brown–Peterson [9] (p. 302-336);
- all torsion in $\mathrm{H}_*(\mathrm{BSpin}; \mathbb{Z})$ and $\mathrm{H}_*(\mathrm{BSpin}; \mathbb{Z})$ is of order 2 (p. 316-317);
- two spin manifolds are spin cobordant if and only if they have the same KO-characteristic numbers and the same Stiefel–Whitney numbers (p. 337);
- two spin^c manifolds are spin^c cobordant if and only if they have the same rational characteristic numbers and the same Stiefel–Whitney numbers (p. 337);
- the ring $(\mathrm{MSpin}^c_* /\text{torsion}) \otimes \mathbb{Z}/2$ is a polynomial algebra over $\mathbb{Z}/2$ on generators y_i, with $|y_i| \in \{2, 4k \mid k \in \mathbb{Z}_{\geq 1}\}$ (p. 347);
- after taking homotopy groups, the forgetful map $\mathrm{MU} \to \mathrm{MSpin}^c$ induces a surjection $\mathrm{MU}_* \to \mathrm{MSpin}^c_* /\text{torsion}$ (p. 348);
- if $\eta : \mathbb{S} \to \mathrm{MSpin}$ is the forgetful map (i.e. the unit), then on homotopy groups, $\pi_0\eta$ is an isomorphism, $\pi_n\eta = 0$ for $n \not\equiv 1, 2 \mod 8$, and $\mathrm{Im}(\pi_n\eta) = \mathbb{Z}/2$, for $n \equiv 1, 2 \mod 8$ (p. 350);

- if $\eta^c : \mathbb{S} \to \mathrm{MSpin}^c$ is the forgetful map (i.e. the unit), then on homotopy groups, $\pi_0\eta^c$ is an isomorphism, $\pi_n\eta^c = 0$ for $n > 0$ (p. 350).

★ [33] 1969. Atiyah–Segal, *Equivariant K-theory and completion.*
Proves the "Atiyah–Segal completion theorem" for all compact Lie groups. The theorem concerns the relationship between the equivariant K-theory, $\mathrm{K}^*_G(X)$, of a G-space, X, and the non-equivariant K-theory of its homotopy quotient (a.k.a. Borel construction), $\mathrm{K}^*(X_G) = \mathrm{K}^*(X \times_G \mathrm{E}G)$. In other words, this is the question of whether or not the G-spectrum K_G is "Borel complete", and identifying the corresponding completion map. Results include:

- **Atiyah–Segal completion theorem:** Let G be a compact Lie group and X be a compact G-space. Assume that $\mathrm{KU}^*_G(X)$ is finite over the representation ring $\mathrm{R}(G) = \mathrm{KU}^0_G(*)$. Let I_G be the augmentation ideal in $\mathrm{R}(G)$, and let

$$\mathrm{KU}^*_G(X)^\wedge_{I_G} := \lim_n \Big(\mathrm{KU}^*_G(X)/(I^n_G \cdot \mathrm{KU}^*_G(X)) \Big).$$

  Then the associated bundle construction induces an isomorphism,

$$\mathrm{KU}^*_G(X)^\wedge_{I_G} \cong \mathrm{KU}^*(X_G).$$

- Specializing to the case $X = *$ yields

$$\mathrm{R}(G)^\wedge_{I_G} \cong \mathrm{KU}^0(\mathrm{B}G).$$

- Proves an analogous result for KR-theory, which has implications for KO-theory as well, though they note that the proof does not work if one works entirely with KO-theory. Recovers the computations of [7] .

I don't know whether the finiteness assumption is necessary or not. In [33] , they indicate situations in which it is satisfied, but I often see the condition omitted in other references. In the case $X = *$ and G finite, the completion theorem was proven in [22] , and for $X = *$ and G connected, it was proven in [28] . Note that these statements and proofs did not involve equivariant K-theory.

★ [37] 1969. Atiyah–Singer, *Index theory for skew-adjoint Fredholm operators.*
Constructs Ω-spectra representing KU and KO out of skew-adjoint Clifford linear Fredholm operators. Includes a proof of Bott periodicity for the spaces in these spectra. The results in this paper can be seen as a generalization of the Atiyah–Jänich theorem (Appendix of [40] , and [114]) from degree 0 K-theory to arbitrary degree.

★ [79] 1970. Donovan–Karoubi, *Graded Brauer groups and K-theory with local coefficients.*

Defines twisted K-theory groups, $\mathrm{KU}^\alpha(X)$ and $\mathrm{KO}^\beta(X)$, for twists,

$$\alpha \in \mathbb{Z}/2 \times \mathrm{H}^1(X;\mathbb{Z}/2) \times \mathrm{H}^3(X;\mathbb{Z})_{\mathrm{tors}},$$

and

$$\beta \in \mathbb{Z}/8 \times \mathrm{H}^1(X;\mathbb{Z}/2) \times \mathrm{H}^2(X;\mathbb{Z}/2),$$

where A_{tors} denotes the torsion subgroup of an abelian group A. (What is the space $K(A,n)_{\mathrm{tors}}$ that classifies $\mathrm{H}^n(-;A)_{\mathrm{tors}}$? Or does it even exist?) The groups classifying the twists α and β over a space X arise from considering the $\mathbb{Z}/2$-graded Brauer groups of bundles of (real or complex) $\mathbb{Z}/2$-graded central simple algebras on X. When X is a point, these groups are given by the Morita classes of Clifford algebras ([26] , [176]). Compare the groups of twists above to the results of [133] ; namely, the twists for K-theory considered here are some, but not all, of the possible twists of K-theory. Generalizes the Thom isomorphism in K-theory ([40] , [26]) to all real vector bundles, where the degree shift must now include non-integral twists. This was the initial motivation for considering the types of twists that they did. Also includes a discussion of the multiplicative structure and Adams operations [1] in twisted K-theory. See [50] for results on the relationship between power operations and Thom classes in the general context of twisted equivariant KR-theory of topological stacks.

★ [39] 1971. Atiyah–Singer, *The index of elliptic operators: V.*
Proves a families index theorem for real elliptic operators. This makes essential use of KR-theory.

★ [84] 1971. Edelson, *Real vector bundles and spaces with free involutions.*
Shows that Real vector bundles (in the sense of [23]) are classified by C_2-equivariant homotopy classes of maps into the infinite complex Grassmannian with its complex conjugation C_2-action. Also studies KR-theory of a free C_2-space, and identifies a classifying space for Real line bundles.

★ [137] 1971. Matumoto, *Equivariant K-theory and Fredholm operators.*
Generalizes the Atiyah–Jänich theorem (Appendix of [40] , and [114]) to the equivariant setting. Results include:

- Let G be a compact Lie group, and let $\mathcal{F}$ be the G-space of Fredholm operators on the G-Hilbert space $L^2(G,\ell^2)$. Then for any compact G-space X, the G-index map induces an isomorphism,

$$\mathrm{index} : [X,\mathcal{F}]_G \xrightarrow{\sim} \mathrm{KU}^0_G(X).$$

For the analogous generalization of [37] in higher degrees, see [FHT?].

★ [166] 1971. Segal, *Equivariant stable homotopy theory.*

Introduces $\mathrm{RO}(G)$-graded equivariant stable homotopy theory for a finite group G, as well as an equivariant J-homomorphism, $J : \mathrm{KO}_G^{-1}(X) \to \mathrm{gl}_1\, \mathbb{S}_G^0(X)$. Identifies the equivariant stable cohomotopy ring of a point (in degree 0) as the Burside ring, $\mathrm{A}(G)$, of the group G:

$$\mathbb{S}_G^0(*) = \pi_0\, \mathbb{S}_G := \operatorname*{colim}_V [S^V, S^V]_G \cong \mathrm{A}(G),$$

where V is a real G-representation, and $\mathrm{A}(G)$ is the Grothendieck group of the category of finite G-sets.

★ [138] 1973. Matumoto, *Equivariant cohomology theories on G-CW copmlexes.*
Develops an Atiyah–Hirzebruch spectral sequence [28] for equivariant cohomology theories, and specifies conditions under which the spectral sequence for KU_G collapses. Applies the spectral sequence to compute KU_G^* of certain types of $\mathrm{O}(n)$-manifolds.

★ [164] 1973. Segal, *The stable homotopy of complex projective space.*
Shows that the inclusion of line bundles into virtual vector bundles,

$$\iota : \mathbb{CP}^\infty = \mathrm{BU}(1) \times \{1\} \to \mathrm{BU} \times \mathbb{Z},$$

induces a split surjection on associated cohomology theories. Specific results:

- Let $P = \Sigma_+^\infty \mathbb{CP}^\infty$. Then for any space X, ι induces a split surjection,

  $$P^0(X) \to \mathrm{KU}^0(X).$$

- There is a homotopy equivalence,

  $$Q(\mathbb{CP}^\infty) = \Omega^\infty P \simeq (\mathrm{BU} \times \mathbb{Z}) \times F,$$

  where the homotopy groups of F are finite.

An analogous result for KO was proven in [47] , and for KR in [150] .

★ [47] 1974. Becker, *Characteristic classes and K-theory.*
Proves an analogue of the results of [164] in the KO case. Results include:

- The inclusion $\mathrm{BO}(2) \to \mathrm{BO}$ induces a split surjection,

  $$(\Sigma_+^\infty \mathrm{BO}(2))^0(X) \to \mathrm{KO}^0(X).$$

See [150] for an analogous result in the KR case.

★ [174] 1974. Thomas, *A relation between K-theory and cohomology.*
Establishes a refinement of the rational isomorphism $\mathrm{KU}^*(X) \otimes \mathbb{Q} \cong \mathrm{H}^{\mathrm{even}}(X; \mathbb{Q})$ of [28] , which includes some torsion, using the theory of λ-rings. Results include:

- For $n \in \mathbb{Z}_{>0}$, let l_n be the set of primes less than n. If X is a CW-complex of dimension at most $4n+3$ and $\mathrm{H}^{4*}(X;\mathbb{Z})$ has no l_{2n+1}-torsion, then there is an isomorphism of abelian groups $\mathrm{KO}(X) \cong \mathrm{H}^{4*}(X;\mathbb{Z})$ modulo 2-torsion.

★ [80] 1977. Douglas, *Extensions of C^*-algebras and K-homology.*
Part of the book [149] . Describes a homology theory, $\mathrm{Ext}_k(X)$, where $\mathrm{Ext}_1(X)$ is a group of equivalence classes of extensions of the C^*-algebra, $\mathcal{K}$, of compact operators on a separable Hilbert space, by the C^*-algebra, $C(X)$, of continuous complex valued functions on X. Explains that Ext_* is the homology theory dual to KU^* by exhibiting a homomorphism $\mathrm{Ext}_1(X) \to \mathrm{Hom}(\mathrm{KU}^1(X), \mathbb{Z})$, using an index construction. Also explains how $\mathrm{Ext}_1(X)$ carries the structure of a $\mathrm{KU}^0(X)$-module.

★ [96] , 1977. Fujii, *On the relation of real cobordism to KR-theory.*
Uses the Real orientation $\mathrm{MU}_{\mathbb{R}} \to \mathrm{KR}$ of [23] to prove a version of the Conner–Floyd isomorphisms [70] for KR-theory. Results include:

- Let (X, A) be a pair of finite C_2-CW complexes. Then $\mathrm{KR}^{0,0}(X, A)$ is a direct summand of $\mathrm{MU}_{\mathbb{R}}^{0,0}(X, A)$.
- There is an isomorphism

$$\mathrm{MU}_{\mathbb{R}}^{\star}(X, A) \otimes_{\mathrm{MU}_{\mathbb{R}}^{\star}} \mathrm{KR}^{\star} \cong \mathrm{KR}^{\star}(X, A),$$

  where $\star$ denotes the $\mathrm{RO}(C_2)$-grading.

★ [133] 1977. Madsen–Snaith–Tornehave, *Infinite loop maps in geometric topology.*
Shows that there are equivalences of infinite loop spaces,

$$\mathrm{GL}_1\,\mathrm{KU} \simeq K(\mathbb{Z}/2, 0) \times K(\mathbb{Z}, 2) \times \mathrm{BSU}_{\otimes},$$

and

$$\mathrm{GL}_1\,\mathrm{KO} \simeq K(\mathbb{Z}/2, 0) \times K(\mathbb{Z}/2, 1) \times \mathrm{BSO}_{\otimes}.$$

These results classify (one connected component of) twists of K-theory by the spaces

$$\mathrm{BGL}_1\,\mathrm{KU} \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times \mathrm{BBSU}_{\otimes},$$

and

$$\mathrm{BGL}_1\,\mathrm{KO} \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2) \times \mathrm{BBSO}_{\otimes}.$$

★ [140] 1977. McDuff, *Configuration spaces.*

Part of the book [149] . Discusses connections between configuration spaces and algebraic K-theory, and uses related technical ideas to propose a new proof of Bott periodicity along the lines of the one in [37] , but replacing the analytic details with some abstract facts about quasifibrations. The idea is essentially a higher dimensional version of the covering space proof that $\pi_1(\mathrm{U}(1)) \cong \mathbb{Z}$, coming from the exponential map $\mathbb{R} \to \mathrm{U}(1)$, where "covering map" is weakened to "quasifibration", "simply connected" is strengthened to "contractible", $\mathrm{U}(1)$ is replaced with U, $\mathbb{Z}$ is replaced with $\mathrm{BU} \times \mathbb{Z}$, and π_1 is replaced with Ω. More precisely, there is a quasifibration $E \to \mathrm{U}$ (given by exponentiation) with fiber $\mathrm{BU} \times \mathbb{Z}$ and E contractible. This proof idea is carried out in [6] in the complex case, and in [49] , together with its addendum, [48] , in the real case.

★ [149] 1977. Morrel–Singer (Eds.), *K-theory and operator algebras.*
A collection of talks from a conference held at the University of Georgia, April 21 – April 25, 1975, on connections between K-theory and operator algebras. Most of the talks are both expository and exploratory in nature. Contains the references: [80] , [140] , [167] .

★ [119] 1978. Karoubi, *K-theory: An introduction.*
A book for learning topological K-theory using the language of Clifford algebras, which handles both the real and complex cases explicitly. Defines K-theory in the setting of Banach categories.

★ [141] 1979. Meier, *Complex and real K-theory and localization.*
Proves that localization with respect to KU and KO coincide (see also [159] and [58] for further analysis of K-local spectra, i.e. spectra that are local with respect to either KU or KO). More specifically, shows that for any CW complex, X, and any abelian group A, $\widetilde{\mathrm{KU}}_*(X;A) = 0$ if and only if $\widetilde{\mathrm{KO}}_*(X;A) = 0$. Applies this result to conclude that the spaces in the KO-theory Ω-spectrum are KU-local, and vice versa. Similarly, Spin as well as any homogeneous space obtained from the stable classical groups $\mathrm{O}, \mathrm{SO}, \mathrm{U}, \mathrm{SU}, \mathrm{Sp}$, and their loop spaces, are K-local. Computes the homotopy groups of the K-localization of $\mathrm{BU}[2n]$ (the $(2n-1)$-connective cover of BU) for $n \geq 2$.

★ [104] 1980. Harris, *Bott periodicity via simplicial spaces.*
Gives a proof of complex Bott periodicity (i.e. that $\Omega\,\mathrm{U} \simeq \mathrm{BU} \times \mathbb{Z}$) that is mostly based on general homotopical facts and properties of unitary matrices. Namely, the diagonalizability of unitary matrices is used to show that

$$\mathrm{B}\Big(\coprod_n \mathrm{BU}(n)\Big) \simeq \mathrm{U},$$

and the group completion theorem [139] is used to show that

$$\mathrm{BU} \times \mathbb{Z} \simeq \Omega\,\mathrm{B}\Big(\coprod_n \mathrm{BU}(n)\Big).$$

There is a similarity between this proof and the ones in [37] and [140] , in that the underlying comparison map is exponentiation. Specifically, suppose $p \in \mathrm{BU}(n)$ is a projection onto a finite dimensional subspace of $\mathbb{C}^\infty$. Then the first equivalence above is induced by the map

$$\exp : \coprod_n \mathrm{BU}(n) \to \Omega\,\mathrm{U}$$

given by $\exp(p) = (t \mapsto e^{2\pi itp} = e^{2\pi it}p + p^\perp)$. It is also noted that the same argument can be applied in the real case to show that

$$\mathrm{B}\Big(\coprod_n \mathrm{BO}(n)\Big) \simeq \mathrm{U}\,/\,\mathrm{O},$$

which gives one of the eight required equivalences for real Bott periodicity, namely,

$$\mathrm{BO} \times \mathbb{Z} \simeq \Omega(\mathrm{U}\,/\,\mathrm{O}).$$

★ [122] 1981. Kasparov, *The operator K-functor and extensions of C^*-algebras.*
Introduces bivariant KK-theory for C^*-algebras, along with its product, which gives KK the structure of an additive category. Works in a general setting which applies to the real, complex, and Real cases, as well as the equivariant setting, for G-C^*-algebras. Proves a version of Bott periodicity for KK which is as general as the one in [25] when applied to commutative C^*-algebras.

★ [150] 1982. Nagata–Nishida–Toda, *Segal–Becker theorem for KR-theory.*
Proves an analogue of the results of [164] and [47] for KR-theory. In fact, the results of this paper recover the previous results for KU and KO by forgetting the C_2-action and by taking C_2-fixed points. Results include:

- Let $\mathbb{CP}^\infty$ be the C_2-space given by complex conjugation and let X be a C_2-CW complex. Then there is a split surjection,

  $$(\Sigma^\infty_+ \mathbb{CP}^\infty)^0(X) \to \mathrm{KR}^0(X),$$

  where $\Sigma^\infty_+ \mathbb{CP}^\infty$ is the suspension spectrum in C_2-spectra, and where the map is induced by viewing a rank 1 Real line bundle as a Real virtual bundle.

★ [72] 1984. Cuntz, *K-theory and C^*-algebras.*
An exposition of C^*-algebraic K-theory. Also includes a proof of a generalization of complex Bott periodicity that applies to a class of functors on C^*-algebras based on a homotopy equivalence between the Toeplitz algebra and the complex numbers.

★ [159] 1984. Ravenel, *Localization with respect to certain periodic homology theories.*

Shows that the Bousfield classes of KO and KU are the same, i.e. they induce the same localization functor on the category of spectra (see [141] for an earlier proof, and [58] for an analysis of the K-local category). Proves the "Smash product theorem", i.e. for $p > 2$ and any spectrum X, we have $L_1 X \simeq X \wedge L_1 \mathbb{S}$, where L_1 denotes localization with respect to p-local K-theory.

★ [158] 1985. Quillen, *Superconnections and the Chern character.*
Introduces the notion of a superconnection on a super ($\mathbb{Z}/2$-graded) vector bundle, and uses it to construct families of differential form representatives for the Chern character. Includes a corresponding geometric description of the Chern character in odd degrees using Clifford linear superconnections. See [135] for a refinement of the Riemann–Roch theorem [27] to an equality of differential forms using this model of the Chern character.

★ [51] 1986. Blackadar, *K-theory for operator algebras.*
A book covering K-theory from the perspective of operator algebras.

★ [135] 1986. Mathai–Quillen, *Superconnections, Thom classes, and equivariant differential forms.*
Proves a refinement of the differentiable Riemann–Roch theorem [27] to an equality of differential form representatives using the Chern character form constructed in [158] and the Thom classes constructed in [26] .

★ [41] 1987. Bahri–Gilkey, *Pin^c cobordism and equivariant Spin^c cobordism of cyclic 2-groups.*
Computes the Pin^c bordism groups, MPin^c_*, as well as $\mathrm{MSpin}^c_*(\mathrm{B}(\mathbb{Z}/2^\nu))$.

★ [73] 1987. Cuntz, *A new look at KK-theory.*
Describes the groups $\mathrm{KK}(A, B)$ in terms of homotopy classes of generalized $*$-homomorphisms from A to B. Specifically, let $qA = \ker(A * A \xrightarrow{m} A)$, where $*$ denotes the coproduct (i.e. free product) of C^*-algebras, and m is the fold map, and let $\mathcal{K}$ be the algebra of compact operators. Then for any (separable?) C^*-algebras A, B,

$$\mathrm{KK}(A, B) \cong \pi_0 \operatorname{Hom}(qA, \mathcal{K} \otimes A),$$

where Hom denotes the space of $*$-homomorphisms.

★ [107] 1987. Higson, *A characterization of KK-theory.*
Shows that KK-theory is the initial additive invariant of C^*-algebras that is stable[3], split exact, and homotopy invariant.

[3] Here, *stable* means invariant under tensoring with the algebra of compact operators on a separable Hilbert space.

★ [163] 1987. Rosenberg–Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor.*
Proves a Künneth theorem and a universal coefficient theorem for Kasparov's KK-theory [122] for C^*-algebras. Let $\mathcal{N}$ be the smallest subcategory of the category of separable nuclear C^*-algebras that satisfies the following properties:

1. $\mathcal{N}$ contains all separable C^*-algebras of Type I,
2. $\mathcal{N}$ is closed under strong Morita equivalence (equivalently, stable isomorphism [60]), inductive limits, extensions, and crossed products by $\mathbb{R}$ and $\mathbb{Z}$,
3. if $J \subset A$ is an ideal, then $A, J \in \mathcal{N} \implies A/J \in \mathcal{N}$, and $A, A/J \in \mathcal{N} \implies J \in \mathcal{N}$.

Let $A \in \mathcal{N}$, and let B be a C^*-algebra with a countable approximate unit. Then

- if $\mathrm{K}_*(B)$ is finitely generated, then there is a short exact sequence,
$$0 \to \mathrm{K}^*(A) \otimes \mathrm{K}_*(B) \to \mathrm{KK}_*(A, B) \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{K}^*(A), \mathrm{K}_*(B)) \to 0,$$
- there is a short exact sequence,
$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \to \mathrm{KK}_*(A, B) \to \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \to 0,$$

where both short exact sequences are natural in each variable and both split unnaturally.

★ [108] 1988. Higson, *Algebraic K-theory of stable C^*-algebras.*
Proves (a conjecture of Karoubi) that for any unital C^*-algebra, A, the algebraic and topological K-theory groups of $A \otimes (\mathcal{B}/\mathcal{K})$ agree, where $\mathcal{B}/\mathcal{K}$ is the Calkin algebra (i.e. $\mathcal{B}$ is the algebra of bounded operators on a separable Hilbert space, and $\mathcal{K}$ is the algebra of compact operators). See also [173] .

★ [123] 1988. Kasparov, *Equivariant KK-theory and the Novikov conjecture.*
A follow-up to Kasparov's work on KK-theory [122] . Applies KK-theory to prove the Novikov conjecture in a certain class of cases. more details. The results first appeared in a 1981 preprint which has since been published as [124] in a modified form.

★ [132] 1989. Lawson–Michelsohn, *Spin geometry.*
A book that introduces the basics of spin geometry, including relations with topological K-theory and applications to topology and geometry. Topics include: a detailed introduction to Clifford algebras and their modules, including the Atiyah–Bott–Shapiro construction, which is extended to the KR case using "Real" Clifford modules; a brief introduction to KU, KO, and KR; spinor bundles and Dirac operators; Thom isomorphisms and the Chern character; various versions of the Atiyah–Singer index theorem; and many applications to topology and differential geometry, including metrics of positive scalar curvature and group actions on

manifolds. Also contains useful appendices on principal bundles, classifying spaces, characteristic classes, and Thom classes.

★ [144] 1989. Miller, *Vector fields on spheres, etc.*
(See [1] .) Notes from a course taught by Haynes Miller at MIT in the fall of 1989 in honor of Frank Adams. They were originally taken by Matthew Ando, and were later transcribed into LaTeX by Eric Peterson, with figures by Yi-Wei Chan, Meng Guo, and Matt Mahowald. The notes were further edited by Michael Donovan and Aaron Mazel-Gee.

★ [58] 1990. Bousfield. *A classification of K-local spectra.*
Classifies spectra which are K-local, i.e. local with respect to either KU or KO (recall that KU and KO have the same Bousfield class [141] , [159]).

★ [151] 1991. Ochanine, *Elliptic genera, modular forms over* KO_**, and the Brown–Kervaire invariant.*
Shows that, in contrast with [70] ,

$$\mathrm{MSU}^*(X) \otimes_{\mathrm{MSU}^*} \mathrm{KO}^* \ncong \mathrm{KO}^*(X).$$

need more here. also need more on what in this paper implies the above result.

★ [111] 1992. Hopkins–Hovey, *Spin cobordism determines real K-theory.*
Proves analogues of the "Conner–Floyd isomorphisms" [70] for spin and spin^c bordism, i.e.

$$(\mathrm{MSpin}^c)^*(X) \otimes_{(\mathrm{MSpin}^c)^*} \mathrm{KU}^* \cong \mathrm{KU}^*(X), \quad \mathrm{MSpin}^*(X) \otimes_{\mathrm{MSpin}^*} \mathrm{KO}^* \cong \mathrm{KO}^*(X);$$

$$\text{or } \mathrm{MSpin}^c_*(X) \otimes_{\mathrm{MSpin}^c_*} \mathrm{KU}_* \cong \mathrm{KU}_*(X), \quad \mathrm{MSpin}_*(X) \otimes_{\mathrm{MSpin}_*} \mathrm{KO}_* \cong \mathrm{KO}_*(X).$$

Note that the proof in this paper does not use a geometric model for K-theory whose (co)cycles are spin(^{c}) manifolds to establish these isomorphisms. Rather, the proof proceeds algebraically by localizing at each prime.

★ [173] 1992. Suslin–Wodzicki, *Excision in algebraic K-theory.*
Shows that all C^*-algebras satisfy (an algebraic notion of) excision in algebraic K-theory and uses this to prove Karoubi's conjecture [cite] that the algebraic and topological K-theory of stable C^*-algebras agree. Results include: Let $\mathrm{K}^{\mathrm{alg}}$ denote algebraic K-theory, and let $\mathrm{K}^{\mathrm{top}}$ denote topological K-theory (i.e. K-theory of C^*-algebras, where the topologies of the algebras and their spaces of automorphisms are taken into account).

- For any C^*-algebra, A,

  $$\mathrm{K}^{\mathrm{top}}(A) \cong \mathrm{K}^{\mathrm{top}}(A \otimes \mathcal{K}) \cong \mathrm{K}^{\mathrm{alg}}(A \otimes \mathcal{K}),$$

  where $\mathcal{K}$ is the algebra of compact operators on an infinite dimensional separable Hilbert space.

Apply this to obtain KU, KO, and maybe KR as *algebraic* K-theory of particular rings.

★ [126] 1993. Kreck–Stolz, *$\mathbb{HP}^2$-bundles and elliptic homology.*
Provides a geometric model of KO-homology as a quotient of spin bordism via the Atiyah–Bott–Shapiro map MSpin → KO [26] . In particular, the kernel of this map is identified explicitly in geometric terms, namely, as the subgroup of MSpin_* consisting of bordism classes containing total spaces of $\mathbb{HP}^2$-bundles. Provides a similar geometric model of elliptic homology.

★ [85] 1995. Fajstrup, *Tate cohomology of periodic K-theory with reality is trivial.*
Shows that the Tate fixed points of KR vanish. This implies that the homotopy and genuine C_2-fixed points of KR are equivalent,

$$\mathrm{KR}^{hC_2} \simeq \mathrm{KR}^{C_2} \simeq \mathrm{KO}\,.$$

★ [124] 1995. Kasparov, *K-theory, group C^*-algebras, and higher signatures (Conspectus) (first distributed 1981).*
A (modified) preprint version of [123] .

★ [177] 1998. Wolbert, *Classifying modules over K-theory spectra.*
Let K denote either KU or KO. Provides an algebraic criterion for determining when a K-local spectrum is a K-module. Shows that any homotopy K-module is equivalent to K-module, and that any ko_*-module can be realized as the homotopy of a ko-module.

★ [6] 1999. Aguilar–Prieto, *Quasifibrations and Bott periodicity.*
Gives a proof of complex Bott periodicity along the lines of the idea sketched out in [140] . For the real case, see [49] and its addendum, [48] .

★ [86] 2000. Freed, *Dirac charge quantization and generalized differential cohomology.*
Discusses physical motivations for generalized differential cohomology. Argues that currents associated to quantized charges for an abelian gauge field should be classified by a generalized cohomology theory equipped with a map to real valued cohomology, $c : E \to \mathrm{H}\mathbb{R}$, together with the additional data of differential forms that are compatible with the map c via de Rham cohomology, in an appropriate sense. See [112] for the underlying mathematical theory of generalized differential cohomology that was being developed at around the same time.

★ [88] 2000. Freed–Hopkins, *On Ramond-Ramond fields and K-theory.*
Builds off of [148] 's observation that Ramond-Ramond fields in type-II string theory have topological invariants in K-theory by suggesting that Ramond-Ramond fields are classified

by *differential* K-theory, thus leading to the development of differential K-theory in the first place.

★ [148] 2000. Moore–Witten, *Self-duality, Ramond-Ramond fields, and K-theory.*
Shows that Ramond-Ramond fields in type-I and type-II string theory determine classes in K-theory. More specifically, Ramond-Ramond fields in type-I string theory on X induce elements of $\mathrm{KO}^{-1}(X)$, Ramond-Ramond fields in type-IIA string theory on X induce elements of $\mathrm{KU}^0(X)$, and Ramond-Ramond fields in type-IIB string theory on X induce elements of $\mathrm{KU}^1(X)$. For a refinement of this idea to differential K-theory, see [88] .

★ [113] 2001. Hu–Kriz, *Real-oriented homotopy theory and an analogue of the Adams–Novikov spectral sequence.*
Shows that the Real orientation of KR, $\mathrm{MU}_{\mathbb{R}} \to \mathrm{KR}$, can be realized as a C_2-$\mathbb{E}_\infty$-map.

★ [115] 2001. Joachim, *A symmetric ring spectrum representing KO-theory.*
Proves that the ring spectrum KO carries the structure of an $\mathbb{E}_\infty$-ring spectrum by constructing a commutative symmetric ring spectrum (i.e. commutative monoid in the symmetric monoidal category of symmetric spectra [134]) which represents KO-theory via Clifford linear Fredholm operators, along the lines of the model of KO in [37] .

★ [49] 2002. Behrens, *A new proof of the Bott periodicity theorem.*
Simplifies the proof of complex Bott periodicity carried out in [6] (following an idea of [140]), and adapts the same proof idea to prove real Bott periodicity. See [48] for corrections and a further simplification of the real case using the language of Clifford algebras.

★ [131] 2003. Laures, *An E_∞ splitting of spin bordism.*
Establishes a multiplicative splitting,

$$\mathrm{MSpin} \simeq T_\zeta \wedge \bigwedge_{i=1}^{\infty} T\,\mathbb{S},$$

where ζ is the generator of $\pi_{-1}L_{K(1)}\,\mathbb{S}$, T_ζ is the $\mathbb{E}_\infty$-cone over ζ, and $T\,\mathbb{S}$ is the free $\mathbb{E}_\infty$-ring generated by the sphere, $\mathbb{S}$. Derives the following consequences:

- The spin bordism ring, MSpin_* is the free θ-algebra[4] over KO_* in (countably) infinitely many generators, and an algorithm is given for constructing the generators.
- Shows that, $K(1)$-locally, the Atiyah–Bott–Shapiro orientation [26] , $\mathrm{MSpin} \to \mathrm{KO}$, refines to an $\mathbb{E}_\infty$-map (see also [116] and [16]).
- Recovers the Conner–Floyd [70] isomorphism of Hopkins–Hovey [111] .

[4] A *θ-algebra* is a commutative algebra over a ring together with a unary operation that satisfies certain properties. The homotopy groups of a $K(1)$-local $\mathbb{E}_\infty$-ring carry the structure of a θ-algebra.

In the process, also shows that the map

$$\mathrm{KO}^0(\mathrm{MSpin}) \to \mathrm{Hom}_{\mathrm{cts}}(\pi_0\,\mathrm{KO} \wedge \mathrm{MSpin}, \pi_0\,\mathrm{KO} \wedge \mathrm{KO}),$$

is injective, i.e. that a map MSpin $\to$ KO is determined by its induced map in KO-homology.

★ [21] 2004. Atiyah–Segal, *Twisted K-theory.*
Constructs a version of twisted KU-theory for twists classified by $K(\mathbb{Z}, 3)$ in terms of families of Fredholm operators on Hilbert bundles

★ [48] 2004. Behrens, *Addendum to "A new proof of the Bott periodicity theorem" [49]* .
An addendum to [49] , correcting mistakes and simplifying the proof of real Bott periodicity using the language of Clifford algebras. This proof can be seen as a variant of the proof in [37] , where the analytic details are replaced with homotopical facts about quasifibrations.

★ [102] 2004. Greenlees, *Equivariant connective K-theory for compact Lie groups.*
Constructs a G-commutative ring spectrum ku_G, for any compact Lie group G, such that

1. the underlying spectrum of ku_G is ku;
2. there is a ring map $\mathrm{MU}_G \to \mathrm{ku}_G$; i.e. ku_G is complex orientable;
3. there is a ring map $\mathrm{ku}_G \to \mathrm{KU}_G$ which is the localization that inverts the Bott element;
4. the coefficient ring ku_G^* is Noetherian, and it's Krull dimension is

$$\max\{2, \text{maximal rank of an abelian subgroup of } G\};$$

5. an analogue of the Atiyah–Segal completion theorem [33] holds, namely,

$$\mathrm{ku}^*(\mathrm{B}G) \cong \mathrm{ku}_G^*(\mathrm{E}G) \cong (\mathrm{ku}_G^*)^\wedge_I,$$

   where $I = \ker(\mathrm{ku}_G^* \to \mathrm{ku}^*)$ is the augmentation ideal.

Also contains partial computations of the coefficient ring ku_G^* for certain groups.

★ [116] 2004. Joachim, *Higher coherences for equivariant K-theory.*
Constructs the spin^c orientation, $\mathrm{MSpin}^c \to \mathrm{KU}$, of [26] as a map of $\mathbb{E}_\infty$-ring spectra, using a C^*-algebra model of K-theory and an orthogonal spectra model of $\mathbb{E}_\infty$-ring spectra. The results are actually obtained in the generality of equivariant K-theory, KU_G, and the methods apply to obtain the spin orientation, MSpin $\to$ KO [26] , as an $\mathbb{E}_\infty$-map as well. The equivariant spin^c orientations constructed here are shown to refine to a map of global orthogonal spectra in [53] .

★ [42] 2005. Baker–Richter, *Invertible modules for commutative $\mathbb{S}$-algebras with residue fields.*
Provides general criteria for when the Picard group of a commutative ring spectrum, R, is isomorphic to the Picard group of its homotopy ring, R_*. Examples include $R \in \{\mathrm{KU}, \mathrm{KO}[\frac{1}{2}], \mathrm{ku}, \mathrm{ko}\}$. In particular, this leads to the fact that $\mathrm{Pic}(\mathrm{KU}) \cong \mathbb{Z}/2$. For a computation of $\mathrm{Pic}(\mathrm{KO})$, see [136] .

★ [43] 2005. Baker–Richter, *On the Γ-cohomology of rings of numerical polynomials and E_∞ structures on K-theory.*
Proves that there are unique $\mathbb{E}_\infty$-structures lifting the usual homotopy ring structures on KU and KO. For the connective versions, see [44] .

★ [81] 2005. Dugger, *An Atiyah–Hirzebruch spectral sequence for KR-theory.*
Develops an analogue of the Atiyah–Hirzebruch spectral sequence [28] in the C_2-equivariant setting of KR-theory, using the equivariant Eilenberg–Mac Lane spectra on the constant Mackey functor, $\underline{\mathbb{Z}}$.

★ [112] 2005. Hopkins–Singer, *Quadratic functions in geometry, topology, and M-theory.*
Constructs differential refinements for arbitrary cohomology theories, and includes a brief discussion about differential K-theory. Includes a discussion of Anderson duality (also see [106]). Add details.

★ [120] 2005. Karoubi, *Bott Periodicity in Topological, Algebraic and Hermitian K-Theory.*
An exposition of the Bott periodicity theorem in various different contexts.

★ [117] 2006. Kandelaki, *Algebraic K-theory of Fredholm modules and KK-theory.*
Expresses $\mathrm{KK}(A, B)$ as the (nonconnective) algebraic K-theory of an additive category of Fredholm (A, B)-bimodules.

★ [44] 2008. Baker–Richter, *Uniqueness of E_∞ structures for connective covers.*
Proves that there are unique $\mathbb{E}_\infty$-structures lifting the usual homotopy ring structures on ku and ko. For the periodic versions, see [43] .

★ [154] 2008. Park, *Complex topological K-theory.*
An introduction to the basics of KU-theory, where proofs are inspired by their generalizations to the context of operator algebras. Includes a brief discussion of Chern–Weil theory.

★ [65] 2009. Bunke–Schick, *Smooth K-theory.*
Constructs a multiplicative differential refinement, $\widehat{\mathrm{KU}}$, of KU-theory using geometric cocycles, together with pushforward maps, $p_! : \widehat{\mathrm{KU}}(W) \to \widehat{\mathrm{KU}}(B)$, associated to KU-orientable proper submersions $p : W \to B$, and refines the Chern character to a natural transformation

$\widehat{\mathrm{ch}} : \widehat{\mathrm{KU}} \to \widehat{\mathrm{H}\mathbb{Q}}$. Proves a refinement of the Atiyah–Singer index theorem [?] regarding the compatibility of pushforwards with the Chern character.

★ [125] 2009. Kitaev, *Periodic table for topological insulators and superconductors.*
Classifies gapped phases of noninteracting fermions in terms of KU-theory and KO-theory using a version of the Atiyah–Bott–Shapiro construction [26] relating Clifford algebras to K-theory as developed in the book [119] .

★ [152] 2009. Ortiz, *Differential Equivariant K-Theory.*
Constructs various models of differential equivariant KU-theory for a finite group.

★ [13] 2010. Ando–Blumberg–Gepner, *Twists of K-theory and TMF.*
Relates the version of twisted K-theory in terms of Fredholm operators [21] to the homotopy theoretic approach developed in [15] , [17] , and [14] . Extends these ideas to construct twists of elliptic cohomology classified by $K(\mathbb{Z}, 4)$.

★ [16] 2010. Ando–Hopkins–Rezk, *Multiplicative orientations of KO-theory and of the spectrum of topological modular forms.*
Computes π_0 of the space of $\mathbb{E}_\infty$-maps from MSpin to KO (which is also equal to π_0 of the space of $\mathbb{E}_\infty$-maps from MString to KO). Results include:

- There is a bijection between $\pi_0\,\mathbb{E}_\infty(\mathrm{MSpin}, \mathrm{KO})$ and the set of even sequences $b : \mathbb{Z}_{\geq 4} \to \mathbb{Q}$, such that $b_k = -\dfrac{B_k}{2k} \mod \mathbb{Z}$, where B_k is the kth Bernoulli number, and for each prime p and each $c \in \mathbb{Z}_p^\times$, the sequence $\{(1-c^k)(1-p^{k-1})b_k\}_{k\geq 4}$ satisfies the "generalized Kummer congruences".
- The sequence in the above classification that corresponds to the Atiyah–Bott–Shapiro orientation ([26] , [116]) is $b_k = -\dfrac{B_k}{2k}$.

The proofs are purely homotopical, and uses the machinery developed in [15] .

★ [61] 2010. Bruner–Greenlees, *Connective real K-theory of finite groups.*
A book about equivariant connective real K-theory, ko_G and kr_G, which contains both a significant amount of exposition, as well as many computations. Includes:

- the $\mathrm{RO}(C_2)$-graded homotopy groups of kr;
- an equivalence $(\Sigma^{-\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma\,\mathrm{ko}$;
- an equivalence $(\Sigma^{-2\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma^{-6}\,\mathrm{ko}\langle 4\rangle$;
- an equivalence $(\Sigma^{-3\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma^{-5}\,\mathrm{ko}\langle 2\rangle$;
- a cofiber sequence $\mathrm{ko} \to \mathrm{kr}^{hC_2} \to \bigvee_{k\geq 1} \Sigma^{-4k}\,\mathrm{H}\,\mathbb{Z}/2$;

- a cofiber sequence $\mathrm{kr}_{hC_2} \to \mathrm{ko} \to \bigvee_{k\geq 0} \Sigma^{4k}\,\mathrm{H}\,\mathbb{Z}/2$;
- the ko homology, $\mathrm{ko}_*(\mathrm{B}G)$, and cohomology, $\mathrm{ko}^*(\mathrm{B}G)$, of the classifying spaces of various groups, including cyclic groups, generalized quaternion groups, the Klein 4-group, dihedral 2-groups, and elementary abelian 2-groups;

★ [62] 2010. Bunke, *Adams operations in smooth K-theory.*
Constructs refinements of the Adams operations [1] for differential (a.k.a. smooth) KU-theory, and proves a corresponding compatibility theorem between the differential Adams operations and pushforward maps for KU-oriented bundles.

★ [66] 2010. Bunke–Schick, *Uniqueness of smooth extensions of generalized cohomology theories.*
Introduces axioms for differential refinements (a.k.a. smooth extensions), $\widehat{E}$, of cohomology theories, E, and proves a corresponding uniqueness theorem for such refinements under certain extra conditions (including the presence of additional structure, such as "integration" or a multiplication). In particular, the differential cohomology theories constructed in [112] satisfy the axioms, and KU and KO satisfy the additional conditions of the uniqueness theorem, implying that any two multiplicative differential refinements of $\widehat{\mathrm{KU}}$ or $\widehat{\mathrm{KO}}$ that satisfy the axioms are isomorphic. They also give examples of non-isomorphic differential refinements of KU-theory in the absence of a multiplicative structure or integration.

★ [93] 2010. Freed–Lott, *An index theorem in differential K-theory.*
Refines the Atiyah–Singer index theorem [?] to differential KU-theory. More specifically, for a proper submersion $\pi : X \to B$ of relative dimension n with a Riemannian structure and a differential spin^c-structure, they construct analytic and topological pushforward maps $\mathrm{ind}^{\mathrm{an}}, \mathrm{ind}^{\mathrm{top}} : \widehat{\mathrm{KU}}^0(X) \to \widehat{\mathrm{KU}}^{-n}(B)$, and show that $\mathrm{ind}^{\mathrm{an}} = \mathrm{ind}^{\mathrm{top}}$.

★ [110] 2010. Hohnhold–Stolz–Teichner, *From minimal geodesics to super symmetric field theories.*
Describes equivalences between eight models for the spaces in (an Ω-spectrum model for) the KO-theory spectrum, including new ones related to supersymmetric Euclidean field theories. These spaces can be roughly describes as follows.

1. Bott's spaces of minimal geodesics [56] .
2. Milnor's spaces of Clifford module structures [145] .
3. Spaces of Clifford-linear unbounded operators.
4. Configuration spaces of finitely many points in a circle labeled by finite dimensional Clifford modules.

5. Spaces of super semigroups of finite rank Clifford-linear operators.

6. Spaces of supersymmetric positive Euclidean field theories of dimension $1|1$.

7. The classifying spaces of certain Quillen-style categories that "categorify" the Atiyah–Bott–Shapiro description of KO_* [26] .

8. Atiyah–Singer's spaces of Clifford-linear Fredholm operators [37] .

Also describes a symmetric ring spectrum representing KO-theory similar to the one in [115] , but using unbounded operators (3) instead of Fredholm operators (8).

★ [71] 2011. Cortiñas, *Algebraic v. topological K-theory: A friendly match.*
An expository chapter emphasizing comparisons between algebraic and topological K-theory of operator algebras, including the proof of Karoubi's conjecture ([cite Karoubi], [173]).

★ [63] 2012. Bunke, *Differential cohomology.*
Lecture notes for a course on differential cohomology. Starts with ordinary differential cohomology, and proceeds to develop generalized differential cohomology (which includes the constructions of [112]) using the machinery of ∞-categories. Includes a detailed discussion of differential KU-theory.

★ [67] 2012. Bunke–Schick, *Differential K-theory: A survey.*
An expository survey of differential KU-theory. Discusses various models, an axiomatic characterization, and analogs of the Riemann–Roch and Atiyah–Singer index theorems for differential KU-theory.

★ [82] 2012. Dugger, *A geometric introduction to K-theory.*
Lecture notes for learning topological K-theory with an algebro-geometric flavor.

★ [18] 2014. Antieau–Gepner–Gómez, *Actions of $K(\pi, n)$ spaces on K-theory and uniqueness of twisted K-theory.*
Proves that

- $[K(\mathbb{Z},3), \mathrm{BGL}_1\,\mathrm{KU}] \cong [K(\mathbb{Z},3), K(\mathbb{Z},3)] \cong \mathbb{Z}$, and
- $[K(\mathbb{Z}/2,2), \mathrm{BGL}_1\,\mathrm{KO}] \cong [K(\mathbb{Z}/2,2), K(\mathbb{Z}/2,2)] \cong \mathbb{Z}/2$.

This allows one to compare various geometric definitions of twisted KU-theory (resp. KO-theory) twisted by elements of $\mathrm{H}^3(-;\mathbb{Z})$ (resp. $\mathrm{H}^2(-;\mathbb{Z}/2)$) via the homotopy class of the corresponding map $K(\mathbb{Z},3) \to \mathrm{BGL}_1\,\mathrm{KU}$ (resp. $K(\mathbb{Z}/2,2) \to \mathrm{BGL}_1\,\mathrm{KO}$). In particular, in the real case, all nontrivial ways of defining twisted KO-theory for twists classified by $K(\mathbb{Z}/2,2)$ are equivalent. Also contains an appendix which explains how to adapt the geometric model for KO-theory given in [115] to the twisted case, within the general framework for twisted cohomology set up in [15] , [14] , and [17] .

★ [53] 2014. Bohmann, *Global orthogonal spectra.*
Gives a definition of global orthogonal spectra, and shows that the equivariant Atiyah–Bott–Shapiro [26] spin^c orientations of KU_G constructed in [116] refine to a map of global orthogonal spectra, $\mathrm{MSpin}^c \to \mathrm{KU}$.

★ [106] 2014. Heard–Stojanoska, *K-theory, reality, and duality.*
Obtains duality results for KO-theory by interpreting it as both the homotopy fixed points and the homotopy orbits of KR. Also includes a discussion of the differentials in the homotopy fixed point spectral sequence for $\mathrm{KR}^{C_2} \simeq \mathrm{KO}$. Results include:

- There is an equivalence, $\mathrm{KR}_{hC_2} = \mathrm{KU}_{hC_2} \simeq \mathrm{KO}$, between C_2-homotopy orbits of KR, i.e. the homotopy orbits of the complex conjugation action on KU, and KO.
- The Anderson dual of KR is equivalent to $\Sigma^4\,\mathrm{KR}$, which implies that the Anderson dual of KO is equivalent to $\Sigma^4\,\mathrm{KO} \simeq \mathrm{KSp}$.
- The $K(1)$-local Brown-Comenetz dual of KO is equivalent to $\Sigma^5\,\mathrm{KO}$.

★ [74] 2015. Dadarlat–Pennig, *Unit spectra of K-theory from strongly self-absorbing C^*-algebras.*
Interprets the group $\mathrm{gl}_1(\mathrm{KU})^1(X)$ as the group of isomorphism classes of locally trivial bundles of C^*-algebras whose fiber is isomorphic to the stable Cuntz algebra, under tensor product. Also provides a similar description of $\mathrm{gl}_1(\mathrm{KU}_{(p)})^1(X)$.

★ [160] 2015. Ricka, *Subalgebras of the $\mathbb{Z}/2$-equivariant Steenrod algebra.*
Computes the C_2-equivariant cohomology of kr, i.e. the connective cover of KR, with coefficients in the constant Mackey functor $\underline{\mathbb{Z}/2}$ as

$$\mathrm{H}^\star(\mathrm{kr}; \underline{\mathbb{Z}/2}) \cong \mathcal{A}^\star/\mathcal{E}^\star(2),$$

where $\mathcal{A}^\star$ is the C_2-equivariant mod 2 Steenrod algebra, and $\mathcal{E}^\star(2)$ is a specified subalgebra.

★ [136] 2016. Mathew–Stojanoska, *The Picard group of topological modular forms via descent theory.*
Computes the Picard group $\mathrm{Pic}(\mathrm{KO}) \cong \mathbb{Z}/8$ using C_2-Galois descent for the C_2-Galois extension $\mathrm{KO} \to \mathrm{KU}$, together with the fact that $\mathrm{Pic}(\mathrm{KU}) \cong \mathrm{Pic}(\mathrm{KU}_*) \cong \mathbb{Z}/2$ (see [42]). The computation uses the homotopy fixed point spectral sequence for $\mathfrak{pic}(\mathrm{KU})$ with C_2-action induced from complex conjugation on KU, where $\mathfrak{pic}$ denotes the Picard spectrum.

★ [155] 2016. Pennig, *A non-commutative model for higher twisted K-theory.*
Models twisted KU-theory for arbitrary twists classified by $\mathrm{BGL}_1(\mathrm{KU})$ in terms of strongly self-absorbing C^*-algebras.

★ [161] 2016. Rosenberg, *Structure and applications of real C^*-algebras.*
An expository survey of real C^*-algebras, including their K-theory. Emphasizes aspects of the theory and applications that are different for the real case. Topics include metrics of positive scalar curvature, orientifold string theory, and the Baum–Connes conjecture.

★ [105] ∼2017. Hatcher, *Vector bundles and K-theory.*
An unfinished introductory text on vector bundles, topological K-theory, and characteristic classes.

★ [142] 2017. Merling, *Equivariant algebraic K-theory of G-rings.*
Introduces a notion of algebraic K-theory for topological G-rings, $\mathrm{K}_G^{\mathrm{top}}$, and shows that

- $\mathrm{K}_G^{\mathrm{top}}(\mathbb{R}, \text{trivial action}) \simeq \mathrm{ko}_G$,
- $\mathrm{K}_G^{\mathrm{top}}(\mathbb{C}, \text{trivial action}) \simeq \mathrm{ku}_G$,
- $\mathrm{K}_{C_2}^{\mathrm{top}}(\mathbb{C}, \text{complex conjugation}) \simeq \mathrm{kr}$,

where the G-rings $\mathbb{R}$ and $\mathbb{C}$ are considered with their topology. Does the same method work to show $\mathrm{K}_{G \rtimes C_2}^{\mathrm{alg}}(\mathbb{C}, \text{trivial} \rtimes \text{complex conjugation}) \simeq \mathrm{kr}_G$, for any Real group G?

★ [178] 2017. Zhang, *A mod 2 index theorem for pin$^-$ manifolds.*
Extends the mod 2 index theorem of Atiyah and Singer [39] to real vector bundles over $(8k+2)$-dimensional compact pin$^-$ manifolds. The analytic index is the reduced η invariant of twisted Dirac operators and the topological index is defined in terms of KO-theory.

★ [128] 2018. Land–Nikolaus, *On the relation between K- and L-theory of C^*-algebras.*
Studies comparison maps between topological K-theory and algebraic L-theory of C^*-algebras. Results include:

- There is a bijection
$$\mathrm{ev}_{\mathbb{C}} : \pi_0 \operatorname{Map}_{\mathrm{Fun}(C^*\mathrm{Alg},\mathrm{Sp})}(\mathrm{ku}, \ell) \to \mathbb{Z}\,.$$
There exists $\tau : \mathrm{ku} \to \ell$ that can be made lax symmetric monoidal, and all such τ have $\mathrm{ev}_{\mathbb{C}}(\tau) = 1$.

- There are equivalences of lax symmetric monoidal functors,
$$\mathrm{KU}[\tfrac{1}{2}] \simeq \mathrm{L}[\tfrac{1}{2}] \quad \text{and} \quad \mathrm{KO}[\tfrac{1}{2}] \simeq \mathrm{L}[\tfrac{1}{2}],$$
where in the first equivalence, the functors are on the category of complex C^*-algebras, and in second equivalence, the functors are on the category of real C^*-algebras.

- In the category of spectra, $[\mathrm{KU}, \mathrm{L}(\mathbb{C})] = [\mathrm{L}(\mathbb{C}), \mathrm{KU}] = [\ell(\mathbb{C}), \mathrm{ku}] = 0$.

★ [19] 2019. Gomez Aparicio–Julg–Valette, *The Baum-Connes conjecture: an extended survey.*
An expository survey of the Baum–Connes conjecture.

★ [147] 2019. Mohsen, *Chern–Simons invariants in KK-theory.*
Realizes the Chern–Simons invariants ([68] , [29] , [30] , [31]) as elements of real KK-theory.

★ [103] 2020. Guillou–Hill–Isaksen–Ravenel, *The cohomology of C_2-equivariant $\mathcal{A}(1)$ and the homotopy of ko_{C_2}.*
Computes the C_2-equivariant mod 2 cohomology of ko_{C_2}, the C_2-equivariant connective cover of KO_{C_2}, and uses it to compute the $\mathrm{RO}(C_2)$-graded homotopy ring of ko_{C_2}. Results include:

- The C_2-fixed point spectrum of ko_{C_2} is $(\mathrm{ko}_{C_2})^{C_2} \simeq \mathrm{ko} \vee \mathrm{ko}$.

- Let σ be the 1-dimensional sign C_2-representation. The C_2-fixed point spectrum of $\Sigma^\sigma \mathrm{ko}_{C_2}$ is $(\Sigma^\sigma \mathrm{ko}_{C_2}) \simeq \mathrm{ko}$.

- The Hopf map induces a C_2-equivariant map $\eta : \Sigma^\sigma \mathrm{ko}_{C_2} \to \mathrm{ko}_{C_2}$, which on C_2-fixed points, induces the map $\eta^{C_2} = (-1, 1) : \mathrm{ko} \to \mathrm{ko} \vee \mathrm{ko}$.

- Complexification $\mathrm{KO}_{C_2} \to \mathrm{KR}$ induces a map on connective covers $c : \mathrm{ko}_{C_2} \to \mathrm{kr}$, which participates in a cofiber sequence,

$$\Sigma^\sigma \mathrm{ko}_{C_2} \xrightarrow{\eta} \mathrm{ko}_{C_2} \xrightarrow{c} \mathrm{kr}\,.$$

  In terms of spectral Mackey functors [cite, also discuss the spectral Mackey functor form of KR somewhere], this cofiber sequence can be written as

$$\begin{tikzcd}
\mathrm{ko} \arrow[r, "{(-1,1)}"] \arrow[d, bend right, "0"'] & \mathrm{ko} \vee \mathrm{ko} \arrow[r, "\nabla"] \arrow[d, bend right, "\nabla"'] & \mathrm{ko} \arrow[d, bend right, "c"'] \\
\Sigma^1 \mathrm{ko} \arrow[r, "\eta"'] \arrow[u, bend right, "\eta"'] \arrow[loop below, "\text{sign}"] & \mathrm{ko} \arrow[r, "c"'] \arrow[u, bend right, "\Delta"'] \arrow[loop below, "\text{trivial}"] & \mathrm{ku}, \arrow[u, bend right, "r"'] \arrow[loop below, "\text{conj}"]
\end{tikzcd}$$

  where $r : \mathrm{ku} \to \mathrm{ko}$ is induced by taking the real vector bundle underlying a complex vector bundle.

- The C_2-equivariant cohomology of ko_{C_2} with coefficients in the constant Mackey functor $\underline{\mathbb{Z}/2}$ is

$$\mathrm{H}^\star(\mathrm{ko}_{C_2}; \underline{\mathbb{Z}/2}) \cong \mathcal{A}^{C_2} /\!/ \mathcal{A}^{C_2}(1),$$

  where $\mathcal{A}^{C_2}$ is the C_2-equivariant mod 2 Steenrod algebra.

- The geometric fixed points of ko_{C_2} is $(\mathrm{ko}_{C_2})^{gC_2} \simeq \bigvee_{k\geq 0} \Sigma^{4k}\mathrm{H}\hat{\mathbb{Z}}_2$, where $\hat{\mathbb{Z}}_2$ is the 2-adic integers.

★ [45] 2022. Balderrama, *K-theory equivariant with respect to an elementary abelian 2-group.*
Computes the $RO(G)$-graded homotopy Mackey functor of KU_G and KO_G for $G = (\mathbb{Z}/2)^n$, together with product structures and power operations.

★ [169] 2022. Stolz, *Positive scalar curvature – constructions and obstructions.*
An expository survey on the state of the question:

> "Which closed connected smooth manifolds of dimension $n \geq 5$ admit a Riemannian metric whose scalar curvature is everywhere positive?"

as of 2022. The question has been resolved for simply connected manifolds, as well as for various other values of the fundamental group. The particular form of the classification, and the class of fundamental groups for which it has been solved, differs for spin and non-spin manifolds. The following are a combination of many prior results by different people.

- Let M be a non-spin manifold whose universal cover is also non-spin. If the fundamental group of M is one of: $e, (\mathbb{Z}/2)^r, (\mathbb{Z}/p)^r$ for p an odd prime and $r < \dim(M)$, then M admits a positive scalar curvature metric.

- Let M be a spin manifold with fundamental group π, let $f : M \to \mathrm{B}\pi$ be the map classifying the universal cover of M, and let $\alpha_\pi : \mathrm{MSpin}_*(\mathrm{B}\pi) \to \mathrm{KO}_*(\mathrm{B}\pi) \to \mathrm{KO}_*(C^*\pi)$ be the composite of the Atiyah–Bott–Shapiro map [26] on $\mathrm{B}\pi$ and the real assembly map. Then for $\pi = e, \mathbb{Z}/2, \mathbb{Z}/(2k+1), (\mathbb{Z}/p)^r$ for p an odd prime and $r > \dim(M)$, or if π has periodic cohomology, M admits a positive scalar curvature metric if and only if $\alpha_\pi([M, f]) = 0 \in \mathrm{KO}_n(C^*\pi)$.

★ [64] 2023. Bunke, *KK- and E-theory via homotopy theory.*
Constructs stable ∞-categories representing KK-theory and E-theory for C^*-algebras from a homotopy theoretic point of view. These constructions are done by localizing the category of separable C^*-algebras. Develops some of the theory of KK and E using only this categorical construction together with basic properties of C^*-algebras, which allows it to serve as an introduction for homotopy theorists. Reworks the proof of Bott periodicity in [72] in the language of ∞-categories.

★ [100] 2023. Gomi–Yamashita, *Differential KO-theory via gradations and mass terms.*
Refines the model of KO-theory developed in [119] in terms of gradations on Clifford modules to a model of differential KO-theory interpreted as "fermionic mass terms". Also develops this model for twisted differential KO-theory.

★ [129] 2023. Land–Nikolaus–Schlichting, *L-theory of C^*-algebras.*
Studies connections between L-theory of real C^*-algebras and topological K-theory (see [128] for previous results along similar lines). Results include:

- There is a unique symmetric monoidal natural transformation (of functors from real C^*-algebras to spectra)
$$\mathrm{ko} \to \mathrm{L}.$$
- For every real C^*-algebra A, the map
$$\mathrm{ko}(A) \otimes_{\mathrm{ko}} \mathrm{L}(\mathbb{R}) \to \mathrm{L}(A)$$
is an equivalence.

★ [50] 2024. Berwick-Evans–Guo, *Power operations preserve Thom classes in twisted equivariant Real K-theory.*
Constructs power operations for twisted KR-theory of topological stacks and shows that these power operations commute with the twisted Atiyah–Bott–Shapiro orientation using algebraic properties of Clifford algebras. Derives corresponding results for KO and KU.

★ [76] 2024. Davis–Wilson, *The cohomology of the connective spectra for K-theory revisited.*
Describes explicit bases for the mod 2 cohomology of connective real and complex K-theory, $\mathrm{H}^*(\mathrm{ko}; \mathbb{Z}/2)$ and $\mathrm{H}^*(\mathrm{ku}; \mathbb{Z}/2)$.

★ [87] 2024. Freed, *Index theory on Pin manifolds.*
An exposition of an index theorem for pin manifolds valued in KO-theory. Also includes a refinement to differential KO-theory, modulo the existence of a proof for the differential KO-index theorem (i.e. the KO-theory version of the main result of [93]). The theorem uses specific inclusions of Pin groups into larger Spin groups to apply the corresponding index theory for spin manifolds. Also includes a motivating discussion explaining the relevance of pin manifolds in physics. States that this is based on work with Mike Hopkins.

★ [75] 2025. Davis, *The connective KO theory of the Eilenberg-MacLane space $K(\mathbb{Z}/2,2)$.*
Computes $\mathrm{ko}_*(K(\mathbb{Z}/2, 2))$ and $\mathrm{ko}^*(K(\mathbb{Z}/2, 2))$, the connective KO-homology and KO-cohomology of the mod 2 Eilenberg–MacLane space, $K(\mathbb{Z}/2, 2)$, using the Adams spectral sequence. Applies the computation to deduce a new result about Stiefel–Whitney classes of Spin manifolds.

★ [101] 2025. Grady, *On the differential K-theory of moduli stacks.*
Computes the connective differential K-theory and the differential cohomology of the moduli stack of principal G-bundles with connection.

★ [162] 2025. Rosenberg, *Complex K-theory of 4-complexes.*
Deduces special properties of $\mathrm{KU}^0(X)$ when X is a 4-dimensional CW complex. Namely,

- for every complex vector bundle V, the class $[V] \in \mathrm{KU}^0(X)$ is determined by $\mathrm{rank}(V) \in \mathrm{H}^0(X;\mathbb{Z})$, its first Chern class $c_1(V) \in \mathrm{H}^2(X;\mathbb{Z})$, and its second Chern class $c_2(V) \in \mathrm{H}^4(X;\mathbb{Z})$;

- for every element $y \in \mathrm{H}^4(X;\mathbb{Z})$, there exists a unique vector bundle V (up to isomorphism) such that $c_1(V) = 0$, and $c_2(V) = y$;

- the ring $\mathrm{KU}^0(X)$ is determined by the ring $\mathrm{H}^{\mathrm{even}}(X;\mathbb{Z})$; specifically, a generators and relations description of $\mathrm{KU}^0(X)$ is provided, where the generators are taken to be elements of $\mathrm{H}^{\mathrm{even}}(X;\mathbb{Z})$.

The analogous underlying results for the real case can be found in [78] , which does not make explicit reference to topological K-theory.

2 Not topological K-theory

This section contains all references cited in this document that do not contain content explicitly about topological K-theory.

★ [20] 1948. Arens–Kaplansky, *Topological representation of algebras.*
Commutative real C^*-algebras are classified by locally compact Hausdorff C_2-spaces via the assignment $(X,\tau) \mapsto C_0(X,\tau) = \{f \in C_0(X) \mid f(\tau x) = \overline{f(x)}\}$.

★ [78] 1959. Dold–Whitney, *Classification of oriented sphere bundles over a 4-complex.*
An analog of [162] in the real case, but it does not discuss the implications for KO-theory explicitly. Work this out.

★ [77] 1962. Dold, *Relation between ordinary and extraordinary homology.*
Extends the isomorphism, $\mathrm{KU}^*(X)\otimes\mathbb{Q} \cong \mathrm{H}^*(X;\mathbb{Q})$, of [28] to arbitrary cohomology theories, e.g. KO. Results include:

- For any $E \in \mathrm{Sp}$, $E^*(X) \otimes \mathbb{Q} \cong \mathrm{H}^*(X; E_* \otimes \mathbb{Q})$.
- For any $E, F \in \mathrm{Sp}$ such that F_* is a $\mathbb{Q}$-vector space, any map of graded abelian groups $\phi : E_* \to F_*$ extends uniquely to a map $\Phi : E^*(\ ) \to F^*(\ )$ of cohomology theories. If ϕ is multiplicative, then Φ is multiplicative.

For KO, the first result implies $\mathrm{KO}^*(X) \otimes \mathbb{Q} \cong \mathrm{H}^*(X; \mathbb{Q}[\alpha^{\pm 1}])$, where $|\alpha| = 4$. The second result implies that the Chern character is determined by its value at a point.

★ [175] 1962. Thomas, *On the cohomology groups of the classifying space for the stable spinor group.*
Computes $\mathrm{H}^*(\mathrm{BSpin}; \mathbb{Z}/2)$ and $\mathrm{H}^*(\mathrm{BSpin}; \mathbb{Z})$.

★ [146] 1963. Milnor, *Spin structures on manifolds.*
Computes some low dimensional spin bordism groups, MSpin_*, which indicates (proves?) that every element of KO_* can be represented by a spin manifold via the Atiyah–Bott–Shapiro orientation [26] .

★ [176] 1964. Wall, *Graded Brauer groups.*
Shows that the set of equivalence classes of $\mathbb{Z}/2$-graded central simple algebras over a field k forms a group, called the *($\mathbb{Z}/2$-) graded Brauer group* (or *Brauer–Wall group*) of k, and shows that the $\mathbb{Z}/2$-graded Brauer group of $\mathbb{R}$ is $\mathbb{Z}/8$ and that of $\mathbb{C}$ is $\mathbb{Z}/2$, in both cases represented by the corresponding type of Clifford algebras under tensor product. The results are obtained for fields k more general than only $\mathbb{R}$ and $\mathbb{C}$.

★ [127] 1965. Kuiper, *The homotopy type of the unitary group of Hilbert space.*

Proves that the group of invertible operators (or unitary operators) on a (real, complex, or quaternionic) separable Hilbert space is contractible. While this result is not about topological K-theory, it is frequently used when constructing analytic models of topological K-theory.

★ [153] 1968. Landweber, *Conjugations on complex manifolds and equivariant homotopy of MU.*
Introduces the Real bordism spectrum $\mathrm{MU}_{\mathbb{R}} = \mathrm{MR}$, which is a C_2-spectrum whose underlying spectrum is complex bordism, MU, and whose C_2-equivariance arises from complex conjugation on $\mathrm{MU}(n) =: \mathrm{MU}_{\mathbb{R}}(n,n)$. By the construction of KR-Thom classes for Real vector bundles in [23] , it follows that there is a ring map $\mathrm{MU}_{\mathbb{R}} \to \mathrm{KR}$. Includes partial results on the homotopy groups $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}}$, including:

- The groups $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}}$ are finitely generated and all torsion is of order 2. The torsion subgroup is the kernel of the homomorphism $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}} \to \pi_{p+q}\,\mathrm{MU}$.
- If $p \geq q$, each element of $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}}$ is represented by a map transverse to $\mathrm{BU}(n) \subset \mathrm{MU}(n)$.
- The homomorphism $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}} \to \pi_q\,\mathrm{MO}$ is surjective if $p \leq q$ and zero if $p > q$ (here the second index, q, denotes the trivial q-dimensional representation).
- If $p \not\equiv q \mod 4$, then $\mathrm{MU}_{\mathbb{R}}$ is finite.
- If $p - q \equiv 4 \mod 8$, then the image of $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}} \to \pi_{p+q}\,\mathrm{MU}$ is $2\pi_{p+q}\,\mathrm{MU}$.
- If $p \equiv q \mod 8$, then the image of $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}} \to \pi_{p+q}\,\mathrm{MU}$ contains $2\pi_{p+q}\,\mathrm{MU} + \mathbb{CP}^1\,\pi_{p+q-2}\,\mathrm{MU}$.
- If $p \equiv q \mod 4$, then $\pi_{p+q}\,\mathrm{MU}_{\mathbb{R}}$ has the same rank as $\pi_{p+q}\,\mathrm{MU}$

★ [157] 1971. Quillen, *The mod 2 cohomology rings of extra-special 2-groups and the spinor groups.*
Computes the rings $\mathrm{H}^*(\mathrm{BSpin}(n);\mathbb{Z}/2)$ and $\mathrm{H}^*(\mathrm{BPin}(n);\mathbb{Z}/2)$, as well as the Stiefel–Whitney classes of spinor representations.

★ [68] 1974. Chern–Simons, *Characteristic forms and geometric invariants.*
Introduces the "Chern–Simons" invariants, which is refined to K-theory in [29] , [30] , and [31] .

★ [60] 1977. Brown–Green–Rieffel, *Stable isomorphism and strong Morita equivalence of C^*-algebras.*
Shows that if A and B are C^*-algebras with countable approximate identities, then A and B are strongly Morita equivalent if and only if they are stably isomorphic. maybe expand these notions in detail here.

★ [98] 1984. Giambalvo–Pengelley, *The homology of MSpin.*
Describes the mod 2 homology of MSpin as a comodule algebra over the dual Steenrod algebra. Includes applications to the structure of the spin bordism ring, MSpin_*.

★ [97] 1994. Getzler, *The equivariant Chern character for non-compact Lie groups.*
Constructs a model for the equivariant Chern character of an equivariant vector bundle along the lines of the Chern–Weil construction in the non-equivariant case.

★ [134] 1999. Hovey–Shipley–Smith, *Symmetric spectra.*
Constructs a symmetric monoidal model category modelling the symmetric monoidal category of spectra under smash product. This is the model in which KO was first proven to be an $\mathbb{E}_\infty$-ring spectrum in [115] .

★ [15] 2008. Ando–Blumberg–Gepner–Hopkins–Rezk, *Units of ring spectra and Thom spectra.*
An older preprint version of [17] and [14] . Develops the theory of units of ring spectra and their relationship with Thom spectra using two approaches: ∞-categories and rigid infinite loop space theory. This theory lays down the theoretical homotopical underpinnings of twisted K-theory. For its application to twisted K-theory, see [13] . This theory is also applied in [16] to identify the space of $\mathbb{E}_\infty$-orientations $\mathrm{MSpin} \to \mathrm{KO}$.

★ [17] 2014. Ando–Blumberg–Gepner–Hopkins–Rezk, *An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology.*
One of two published versions of [15] developing the theory of units of ring spectra and their relationship with Thom spectra via ∞-category theory. This theory lays down the theoretical homotopical underpinnings of twisted K-theory. For its application to twisted K-theory, see [13] .

★ [14] 2014. Ando–Blumberg–Gepner–Hopkins–Rezk, *Units of ring spectra, orientations, and Thom spectra via rigid infinite loop space theory.*
One of two published versions of [15] developing the theory of units of ring spectra and their relationship with Thom spectra via rigid infinite loop space theory. This theory lays down the theoretical homotopical underpinnings of twisted K-theory. For its application to twisted K-theory, see [13] .

3 Expository references

This section contains references that are either mostly expository in nature, or select research papers that cover important parts of the theory and are still good references for learning the material that they cover.

★ [28] 1961. Atiyah–Hirzebruch, *Vector bundles and homogeneous spaces.*
Develops the basic properties of K-theory as a cohomology theory. Introduces the Atiyah–Hirzebruch spectral sequence. Written in terms of KU, but many constructions carry over to KO. Shows that the Chern character induces an isomorphism,

$$\mathrm{KU}^*(X) \otimes \mathbb{Q} \cong \mathrm{H}^*(X; \mathbb{Q}),$$

where the $*$ represents a $\mathbb{Z}/2$-grading on both sides (on the left side, this is induced by Bott periodicity, and on the right side, it is induced by the $\mathbb{Z}$-grading via the quotient $\mathbb{Z} \to \mathbb{Z}/2$). In particular, $\mathrm{KU}^0(X) \otimes \mathbb{Q} \cong \mathrm{H}^{\mathrm{even}}(X; \mathbb{Q})$ and $\mathrm{KU}^1(X) \otimes \mathbb{Q} \cong \mathrm{H}^{\mathrm{odd}}(X; \mathbb{Q})$. Also proves a special case of the Atiyah–Segal completion theorem [33] , namely when $X = *$ and G is a compact connected Lie group, relating the representation ring of G to $\mathrm{KU}^*(\mathrm{B}G)$.
See [174] for a refinement of the relation between K-theory and ordinary cohomology that includes torsion and an application to KO-theory. See [77] for results on generalizations of the Chern character to arbitrary cohomology theories.

★ [40] 1964. Atiyah, *K-theory.*
A book that covers the basics of complex K-theory (KU-theory) without assuming knowledge of algebraic topology. The real case (KO) is not discussed, but it is a very useful text for understanding the ideas of both theories. Includes a proof of complex Bott periodicity, and a proof of the Thom isomorphism in KU-theory for complex vector bundles, inducing a complex orientation $\mathrm{MU} \to \mathrm{KU}$. See [26] and [23] for the Thom isomorphisms for spin, spin^c, and Real vector bundles, and see [25] for the Thom isomorphisms for Real spin^c vector bundles, as well as for equivariant versions of all of the above Thom isomorphisms. Also contains an appendix proving (independently from [114]) the "Atiyah–Jänich theorem", namely, that the index induces an isomorphism,

$$\mathrm{index} : [X, \mathcal{F}] \to \mathrm{KU}^0(X),$$

where $\mathcal{F}$ is the space of Fredholm operators on a separable complex Hilbert space.

★ [165] 1968. Segal, *Equivariant K-theory.*
An introduction to equivariant K-theory built from G-equivariant vector bundles. This work is Segal's thesis, and much of it is attributed to Atiyah.

★ [172] 1968. Stong, *Notes on cobordism theory.*

An exposition of many topics in cobordism theory, including detailed treatments of various relationships between spin and spin^c bordism with topological K-theory (Chapter XI). The following link contains a detailed table of contents which was compiled and made available by Landweber and Ravenel. Contents include:

- a computation of $\mathrm{H}^*(\mathrm{BSpin}; \mathbb{Z}/2)$ (p. 291-292) and $\mathrm{H}^*(\mathrm{BSpin}^c; \mathbb{Z}/2)$ (p. 293);
- if a vector bundle admits a KO-orientation, then it must admit a spin structure (p. 301);
- an exposition of the main results of Anderson–Brown–Peterson [9] (p. 302-336);
- all torsion in $\mathrm{H}_*(\mathrm{BSpin}; \mathbb{Z})$ and $\mathrm{H}_*(\mathrm{BSpin}; \mathbb{Z})$ is of order 2 (p. 316-317);
- two spin manifolds are spin cobordant if and only if they have the same KO-characteristic numbers and the same Stiefel–Whitney numbers (p. 337);
- two spin^c manifolds are spin^c cobordant if and only if they have the same rational characteristic numbers and the same Stiefel–Whitney numbers (p. 337);
- the ring $(\mathrm{MSpin}^c_* /\mathrm{torsion}) \otimes \mathbb{Z}/2$ is a polynomial algebra over $\mathbb{Z}/2$ on generators y_i, with $|y_i| \in \{2, 4k \mid k \in \mathbb{Z}_{\geq 1}\}$ (p. 347);
- after taking homotopy groups, the forgetful map $\mathrm{MU} \to \mathrm{MSpin}^c$ induces a surjection $\mathrm{MU}_* \to \mathrm{MSpin}^c_* /\mathrm{torsion}$ (p. 348);
- if $\eta : \mathbb{S} \to \mathrm{MSpin}$ is the forgetful map (i.e. the unit), then on homotopy groups, $\pi_0\eta$ is an isomorphism, $\pi_n\eta = 0$ for $n \not\equiv 1, 2 \mod 8$, and $\mathrm{Im}(\pi_n\eta) = \mathbb{Z}/2$, for $n \equiv 1, 2 \mod 8$ (p. 350);
- if $\eta^c : \mathbb{S} \to \mathrm{MSpin}^c$ is the forgetful map (i.e. the unit), then on homotopy groups, $\pi_0\eta^c$ is an isomorphism, $\pi_n\eta^c = 0$ for $n > 0$ (p. 350).

★ [80] 1977. Douglas, *Extensions of C^*-algebras and K-homology.*
Part of the book [149] . Describes a homology theory, $\mathrm{Ext}_k(X)$, where $\mathrm{Ext}_1(X)$ is a group of equivalence classes of extensions of the C^*-algebra, $\mathcal{K}$, of compact operators on a separable Hilbert space, by the C^*-algebra, $C(X)$, of continuous complex valued functions on X. Explains that Ext_* is the homology theory dual to KU^* by exhibiting a homomorphism $\mathrm{Ext}_1(X) \to \mathrm{Hom}(\mathrm{KU}^1(X), \mathbb{Z})$, using an index construction. Also explains how $\mathrm{Ext}_1(X)$ carries the structure of a $\mathrm{KU}^0(X)$-module.

★ [140] 1977. McDuff, *Configuration spaces.*

Part of the book [149] . Discusses connections between configuration spaces and algebraic K-theory, and uses related technical ideas to propose a new proof of Bott periodicity along the lines of the one in [37] , but replacing the analytic details with some abstract facts about quasifibrations. The idea is essentially a higher dimensional version of the covering space proof that $\pi_1(\mathrm{U}(1)) \cong \mathbb{Z}$, coming from the exponential map $\mathbb{R} \to \mathrm{U}(1)$, where "covering map" is weakened to "quasifibration", "simply connected" is strengthened to "contractible", $\mathrm{U}(1)$ is replaced with U, $\mathbb{Z}$ is replaced with $\mathrm{BU} \times \mathbb{Z}$, and π_1 is replaced with Ω. More precisely, there is a quasifibration $E \to \mathrm{U}$ (given by exponentiation) with fiber $\mathrm{BU} \times \mathbb{Z}$ and E contractible. This proof idea is carried out in [6] in the complex case, and in [49] , together with its addendum, [48] , in the real case.

★ [149] 1977. Morrel–Singer (Eds.), *K-theory and operator algebras.*
A collection of talks from a conference held at the University of Georgia, April 21 – April 25, 1975, on connections between K-theory and operator algebras. Most of the talks are both expository and exploratory in nature. Contains the references: [80] , [140] , [167] .

★ [119] 1978. Karoubi, *K-theory: An introduction.*
A book for learning topological K-theory using the language of Clifford algebras, which handles both the real and complex cases explicitly. Defines K-theory in the setting of Banach categories.

★ [72] 1984. Cuntz, *K-theory and C^*-algebras.*
An exposition of C^*-algebraic K-theory. Also includes a proof of a generalization of complex Bott periodicity that applies to a class of functors on C^*-algebras based on a homotopy equivalence between the Toeplitz algebra and the complex numbers.

★ [51] 1986. Blackadar, *K-theory for operator algebras.*
A book covering K-theory from the perspective of operator algebras.

★ [132] 1989. Lawson–Michelsohn, *Spin geometry.*
A book that introduces the basics of spin geometry, including relations with topological K-theory and applications to topology and geometry. Topics include: a detailed introduction to Clifford algebras and their modules, including the Atiyah–Bott–Shapiro construction, which is extended to the KR case using "Real" Clifford modules; a brief introduction to KU, KO, and KR; spinor bundles and Dirac operators; Thom isomorphisms and the Chern character; various versions of the Atiyah–Singer index theorem; and many applications to topology and differential geometry, including metrics of positive scalar curvature and group actions on manifolds. Also contains useful appendices on principal bundles, classifying spaces, characteristic classes, and Thom classes.

★ [124] 1995. Kasparov, *K-theory, group C^*-algebras, and higher signatures (Conspectus)*

(first distributed 1981).
A (modified) preprint version of [123] .

★ [120] 2005. Karoubi, *Bott Periodicity in Topological, Algebraic and Hermitian K-Theory.*
An exposition of the Bott periodicity theorem in various different contexts.

★ [154] 2008. Park, *Complex topological K-theory.*
An introduction to the basics of KU-theory, where proofs are inspired by their generalizations to the context of operator algebras. Includes a brief discussion of Chern–Weil theory.

★ [61] 2010. Bruner–Greenlees, *Connective real K-theory of finite groups.*
A book about equivariant connective real K-theory, ko_G and kr_G, which contains both a significant amount of exposition, as well as many computations. Includes:

- the $\mathrm{RO}(C_2)$-graded homotopy groups of kr;
- an equivalence $(\Sigma^{-\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma\,\mathrm{ko}$;
- an equivalence $(\Sigma^{-2\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma^{-6}\,\mathrm{ko}\langle 4\rangle$;
- an equivalence $(\Sigma^{-3\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma^{-5}\,\mathrm{ko}\langle 2\rangle$;
- a cofiber sequence $\mathrm{ko} \to \mathrm{kr}^{hC_2} \to \bigvee_{k\geq 1} \Sigma^{-4k}\,\mathrm{H}\,\mathbb{Z}/2$;
- a cofiber sequence $\mathrm{kr}_{hC_2} \to \mathrm{ko} \to \bigvee_{k\geq 0} \Sigma^{4k}\,\mathrm{H}\,\mathbb{Z}/2$;
- the ko homology, $\mathrm{ko}_*(\mathrm{B}G)$, and cohomology, $\mathrm{ko}^*(\mathrm{B}G)$, of the classifying spaces of various groups, including cyclic groups, generalized quaternion groups, the Klein 4-group, dihedral 2-groups, and elementary abelian 2-groups;

★ [71] 2011. Cortiñas, *Algebraic v. topological K-theory: A friendly match.*
An expository chapter emphasizing comparisons between algebraic and topological K-theory of operator algebras, including the proof of Karoubi's conjecture ([cite Karoubi], [173]).

★ [63] 2012. Bunke, *Differential cohomology.*
Lecture notes for a course on differential cohomology. Starts with ordinary differential cohomology, and proceeds to develop generalized differential cohomology (which includes the constructions of [112]) using the machinery of ∞-categories. Includes a detailed discussion of differential KU-theory.

★ [67] 2012. Bunke–Schick, *Differential K-theory: A survey.*

An expository survey of differential KU-theory. Discusses various models, an axiomatic characterization, and analogs of the Riemann–Roch and Atiyah–Singer index theorems for differential KU-theory.

★ [82] 2012. Dugger, *A geometric introduction to K-theory.*
Lecture notes for learning topological K-theory with an algebro-geometric flavor.

★ [161] 2016. Rosenberg, *Structure and applications of real C^*-algebras.*
An expository survey of real C^*-algebras, including their K-theory. Emphasizes aspects of the theory and applications that are different for the real case. Topics include metrics of positive scalar curvature, orientifold string theory, and the Baum–Connes conjecture.

★ [105] ∼2017. Hatcher, *Vector bundles and K-theory.*
An unfinished introductory text on vector bundles, topological K-theory, and characteristic classes.

★ [156] 2018. Peterson, *Formal geometry and bordism operations.*
A book emphasizing an algebro-geometric approach to chromatic homotopy theory. Contains an exposition of [16] 's homotopical construction of an $\mathbb{E}_\infty$-map MSpin $\to$ KO.

★ [19] 2019. Gomez Aparicio–Julg–Valette, *The Baum-Connes conjecture: an extended survey.*
An expository survey of the Baum–Connes conjecture.

★ [169] 2022. Stolz, *Positive scalar curvature – constructions and obstructions.*
An expository survey on the state of the question:

> "Which closed connected smooth manifolds of dimension $n \geq 5$ admit a Riemannian metric whose scalar curvature is everywhere positive?"

as of 2022. The question has been resolved for simply connected manifolds, as well as for various other values of the fundamental group. The particular form of the classification, and the class of fundamental groups for which it has been solved, differs for spin and non-spin manifolds. The following are a combination of many prior results by different people.

- Let M be a non-spin manifold whose universal cover is also non-spin. If the fundamental group of M is one of: $e, (\mathbb{Z}/2)^r, (\mathbb{Z}/p)^r$ for p an odd prime and $r < \dim(M)$, then M admits a positive scalar curvature metric.
- Let M be a spin manifold with fundamental group π, let $f : M \to \mathrm{B}\pi$ be the map classifying the universal cover of M, and let $\alpha_\pi : \mathrm{MSpin}_*(\mathrm{B}\pi) \to \mathrm{KO}_*(\mathrm{B}\pi) \to \mathrm{KO}_*(C^*\pi)$ be the composite of the Atiyah–Bott–Shapiro map [26] on $\mathrm{B}\pi$ and the real assembly map. Then for $\pi = e, \mathbb{Z}/2, \mathbb{Z}/(2k+1), (\mathbb{Z}/p)^r$ for p an odd prime and $r > \dim(M)$, or

if π has periodic cohomology, M admits a positive scalar curvature metric if and only if $\alpha_\pi([M,f]) = 0 \in \mathrm{KO}_n(C^*\pi)$.

★ [64] 2023. Bunke, *KK- and E-theory via homotopy theory.*
Constructs stable ∞-categories representing KK-theory and E-theory for C^*-algebras from a homotopy theoretic point of view. These constructions are done by localizing the category of separable C^*-algebras. Develops some of the theory of KK and E using only this categorical construction together with basic properties of C^*-algebras, which allows it to serve as an introduction for homotopy theorists. Reworks the proof of Bott periodicity in [72] in the language of ∞-categories.

★ [87] 2024. Freed, *Index theory on Pin manifolds.*
An exposition of an index theorem for pin manifolds valued in KO-theory. Also includes a refinement to differential KO-theory, modulo the existence of a proof for the differential KO-index theorem (i.e. the KO-theory version of the main result of [93]). The theorem uses specific inclusions of Pin groups into larger Spin groups to apply the corresponding index theory for spin manifolds. Also includes a motivating discussion explaining the relevance of pin manifolds in physics. States that this is based on work with Mike Hopkins.

4 Proofs of Bott periodicity

This section contains references that prove Bott periodicity. This includes references that propose a new proof idea or sketch, as well as references that reformulate a previous proof in a significant or more expository way. The threshold of which references satisfy these criteria is far from objective, so feedback that can improve the quality of this section is much appreciated.

★ [56] 1957. Bott, *The stable homotopy of the classical groups.*
Proves Bott periodicity for the first time, using Morse theory.

★ [57] 1959. Bott, *Quelques remarques sur les théorèmes de périodicité.*
Proves "strong" Bott periodicity, i.e. that the periodicity in KU- or KO-cohomology is induced by multiplication by a particular element. Note that the standard Bott periodicity theorem about the homotopy types of $\Omega^k \mathrm{U}$ and $\Omega^k \mathrm{O}$ don't depend on any multiplicative structure of the associated spectra, KU and KO. It turns out that the periodicity theorem is, in fact, related to the natural multiplicative structures that the K-theory spectra carry.

★ [145] 1963. Milnor, *Morse theory.*
Contains proofs of both complex and real Bott periodicity by identifying spaces of minimal geodesics in orthogonal groups with spaces of Clifford module structures. The Clifford language is not used explicitly; instead, the language of "commuting complex structures" is used, where a complex structure is the same as an action by a Clifford algebra generator.

★ [40] 1964. Atiyah, *K-theory.*
A book that covers the basics of complex K-theory (KU-theory) without assuming knowledge of algebraic topology. The real case (KO) is not discussed, but it is a very useful text for understanding the ideas of both theories. Includes a proof of complex Bott periodicity, and a proof of the Thom isomorphism in KU-theory for complex vector bundles, inducing a complex orientation $\mathrm{MU} \to \mathrm{KU}$. See [26] and [23] for the Thom isomorphisms for spin, spin^c, and Real vector bundles, and see [25] for the Thom isomorphisms for Real spin^c vector bundles, as well as for equivariant versions of all of the above Thom isomorphisms. Also contains an appendix proving (independently from [114]) the "Atiyah–Jänich theorem", namely, that the index induces an isomorphism,

$$\text{index} : [X, \mathcal{F}] \to \mathrm{KU}^0(X),$$

where $\mathcal{F}$ is the space of Fredholm operators on a separable complex Hilbert space.

★ [23] 1966. Atiyah, *K-theory and reality.*
Introduces KR as a C_2-equivariant cohomology theory, sometimes called "Real K-theory", built from complex vector bundles with Real structure, that restricts to KO-theory on the

subcategory of C_2-spaces with trivial C_2-action. Shows that as an $\mathrm{RO}(C_2)$-graded cohomology theory, KR is $(1,1)$-periodic (i.e. $\mathbb{C}$-periodic). More generally, shows that KR has Thom isomorphisms for Real vector bundles, i.e. complex vector bundles with Real structure. This means that KR has a Real orientation, i.e. a ring map, $\mathrm{MU}_{\mathbb{R}} \to \mathrm{KR}$, from the Real bordism spectrum introduced in [153] .

★ [25] 1968. Atiyah, *Bott periodicity and the index of elliptic operators.*
Uses index theory to prove generalizations of the 8-fold Bott periodicity of KO, the 2-fold Bott periodicity of KU, and the $(1,1)$-fold Bott periodicity of KR in the equivariant settings. The most general statement from which all others follow is the following.

- Let $G \to \mathrm{Spin}^c(p,q)$ be a C_2-equivariant homomorphism of groups with C_2-action, and suppose $p \equiv q \mod 8$. Then there is a class $u \in \mathrm{KR}_G(\mathbb{R}^{p,q})$, such that multiplication by u induces an isomorphism,

  $$\mathrm{KR}_G(X) \cong \mathrm{KR}_G(\mathbb{R}^{p,q} \times X).$$

  Here, C_2 acts on $\mathbb{R}^{p,q} = \mathbb{R}^{p+q}$ trivially on the $\mathbb{R}^p$ factor and by -1 on the $\mathbb{R}^q$ factor, and $\mathrm{Spin}^c(p,q)$ is the spin group in the Real Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{R}^{p,q})$ with the induced C_2-action.

The corresponding Thom isomorphisms are formal consequences of the equivariant periodicity theorems.

★ [37] 1969. Atiyah–Singer, *Index theory for skew-adjoint Fredholm operators.*
Constructs Ω-spectra representing KU and KO out of skew-adjoint Clifford linear Fredholm operators. Includes a proof of Bott periodicity for the spaces in these spectra. The results in this paper can be seen as a generalization of the Atiyah–Jänich theorem (Appendix of [40] , and [114]) from degree 0 K-theory to arbitrary degree.

★ [140] 1977. McDuff, *Configuration spaces.*
Part of the book [149] . Discusses connections between configuration spaces and algebraic K-theory, and uses related technical ideas to propose a new proof of Bott periodicity along the lines of the one in [37] , but replacing the analytic details with some abstract facts about quasifibrations. The idea is essentially a higher dimensional version of the covering space proof that $\pi_1(\mathrm{U}(1)) \cong \mathbb{Z}$, coming from the exponential map $\mathbb{R} \to \mathrm{U}(1)$, where "covering map" is weakened to "quasifibration", "simply connected" is strengthened to "contractible", $\mathrm{U}(1)$ is replaced with U, $\mathbb{Z}$ is replaced with $\mathrm{BU} \times \mathbb{Z}$, and π_1 is replaced with Ω. More precisely, there is a quasifibration $E \to \mathrm{U}$ (given by exponentiation) with fiber $\mathrm{BU} \times \mathbb{Z}$ and E contractible. This proof idea is carried out in [6] in the complex case, and in [49] , together with its addendum, [48] , in the real case.

★ [104] 1980. Harris, *Bott periodicity via simplicial spaces.*

Gives a proof of complex Bott periodicity (i.e. that $\Omega\,\mathrm{U} \simeq \mathrm{BU} \times \mathbb{Z}$) that is mostly based on general homotopical facts and properties of unitary matrices. Namely, the diagonalizability of unitary matrices is used to show that

$$\mathrm{B}\Big(\coprod_n \mathrm{BU}(n)\Big) \simeq \mathrm{U},$$

and the group completion theorem [139] is used to show that

$$\mathrm{BU} \times \mathbb{Z} \simeq \Omega\,\mathrm{B}\Big(\coprod_n \mathrm{BU}(n)\Big).$$

There is a similarity between this proof and the ones in [37] and [140] , in that the underlying comparison map is exponentiation. Specifically, suppose $p \in \mathrm{BU}(n)$ is a projection onto a finite dimensional subspace of $\mathbb{C}^\infty$. Then the first equivalence above is induced by the map

$$\exp : \coprod_n \mathrm{BU}(n) \to \Omega\,\mathrm{U}$$

given by $\exp(p) = (t \mapsto e^{2\pi itp} = e^{2\pi it}p + p^\perp)$. It is also noted that the same argument can be applied in the real case to show that

$$\mathrm{B}\Big(\coprod_n \mathrm{BO}(n)\Big) \simeq \mathrm{U}\,/\,\mathrm{O},$$

which gives one of the eight required equivalences for real Bott periodicity, namely,

$$\mathrm{BO} \times \mathbb{Z} \simeq \Omega(\mathrm{U}\,/\,\mathrm{O}).$$

★ [122] 1981. Kasparov, *The operator K-functor and extensions of C^*-algebras.*
Introduces bivariant KK-theory for C^*-algebras, along with its product, which gives KK the structure of an additive category. Works in a general setting which applies to the real, complex, and Real cases, as well as the equivariant setting, for G-C^*-algebras. Proves a version of Bott periodicity for KK which is as general as the one in [25] when applied to commutative C^*-algebras.

★ [72] 1984. Cuntz, *K-theory and C^*-algebras.*
An exposition of C^*-algebraic K-theory. Also includes a proof of a generalization of complex Bott periodicity that applies to a class of functors on C^*-algebras based on a homotopy equivalence between the Toeplitz algebra and the complex numbers.

★ [6] 1999. Aguilar–Prieto, *Quasifibrations and Bott periodicity.*
Gives a proof of complex Bott periodicity along the lines of the idea sketched out in [140] . For the real case, see [49] and its addendum, [48] .

★ [49] 2002. Behrens, *A new proof of the Bott periodicity theorem.*
Simplifies the proof of complex Bott periodicity carried out in [6] (following an idea of [140]), and adapts the same proof idea to prove real Bott periodicity. See [48] for corrections and a further simplification of the real case using the language of Clifford algebras.

★ [48] 2004. Behrens, *Addendum to "A new proof of the Bott periodicity theorem" [49]* .
An addendum to [49] , correcting mistakes and simplifying the proof of real Bott periodicity using the language of Clifford algebras. This proof can be seen as a variant of the proof in [37] , where the analytic details are replaced with homotopical facts about quasifibrations.

★ [64] 2023. Bunke, *KK- and E-theory via homotopy theory.*
Constructs stable ∞-categories representing KK-theory and E-theory for C^*-algebras from a homotopy theoretic point of view. These constructions are done by localizing the category of separable C^*-algebras. Develops some of the theory of KK and E using only this categorical construction together with basic properties of C^*-algebras, which allows it to serve as an introduction for homotopy theorists. Reworks the proof of Bott periodicity in [72] in the language of ∞-categories.

5 KO-theory

The references singled out in this section either use or prove things about the particularities of KO-theory in some way (as opposed to KU-theory).

★ [56] 1957. Bott, *The stable homotopy of the classical groups.*
Proves Bott periodicity for the first time, using Morse theory.

★ [57] 1959. Bott, *Quelques remarques sur les théorèmes de périodicité.*
Proves "strong" Bott periodicity, i.e. that the periodicity in KU- or KO-cohomology is induced by multiplication by a particular element. Note that the standard Bott periodicity theorem about the homotopy types of $\Omega^k \mathrm{U}$ and $\Omega^k \mathrm{O}$ don't depend on any multiplicative structure of the associated spectra, KU and KO. It turns out that the periodicity theorem is, in fact, related to the natural multiplicative structures that the K-theory spectra carry.

★ [1] 1962. Adams, *Vector fields on spheres.*
Introduces certain power operations in KU-theory and KO-theory (now called "Adams operations") and uses them to determine the maximum number of linearly independent vector fields on each sphere S^n. Notably, the computations involving KO-theory are essential to the proof; consideration of KU-theory alone is insufficient. Also computes stuff about projective spaces. See [24] for a more thorough analysis of power operations in K-theory from the perspective of equivariant K-theory. write some details out here.

★ [145] 1963. Milnor, *Morse theory.*
Contains proofs of both complex and real Bott periodicity by identifying spaces of minimal geodesics in orthogonal groups with spaces of Clifford module structures. The Clifford language is not used explicitly; instead, the language of "commuting complex structures" is used, where a complex structure is the same as an action by a Clifford algebra generator.

★ [146] 1963. Milnor, *Spin structures on manifolds.*
Computes some low dimensional spin bordism groups, MSpin_*, which indicates (proves?) that every element of KO_* can be represented by a spin manifold via the Atiyah–Bott–Shapiro orientation [26] .

★ [7] 1964. Anderson, *The real K-theory of classifying spaces.*

Computes $\mathrm{KO}^*(\mathrm{B}G)$ for a compact connected Lie group, G.

$$\mathrm{KO}^i(\mathrm{B}G) \cong \begin{cases} \mathrm{RO}(G)^\wedge, & i=0 \\ 0, & i=1 \\ \mathrm{RU}(G)^\wedge / \mathrm{RO}(G)^\wedge, & i=2 \\ \mathrm{RSp}(G)^\wedge / \mathrm{RU}(G)^\wedge, & i=3 \\ \mathrm{RSp}(G)^\wedge \cong \mathrm{KSp}^0(\mathrm{B}G), & i=4 \\ 0, & i=5 \\ \mathrm{RU}(G)^\wedge / \mathrm{RSp}(G)^\wedge, & i=6 \\ \mathrm{RO}(G)^\wedge / \mathrm{RU}(G)^\wedge, & i=7. \end{cases} \quad \mathrm{mod}\ 8$$

These computations also follow from [33] .

★ [26] 1964. Atiyah–Bott–Shapiro, *Clifford modules.*
Develops the basic algebraic features of Clifford algebras and modules that are relevant for topological K-theory, especially for treating the KO and KU cases uniformly. In particular, they establish an isomorphism between the graded ring KO_* (KU_*) and a graded ring of equivalence classes[5] of finite-dimensional virtual real (complex) Clifford modules, under direct sum and tensor product of modules. They realize this construction more generally to construct Thom classes in KO-theory (KU-theory) for vector bundles with a spin (spin^c) structure, which results in a multiplicative map of ring spectra, $\mathrm{MSpin} \to \mathrm{KO}$ ($\mathrm{MSpin}^c \to \mathrm{KU}$). After taking homotopy groups, this map recovers the KO_*-valued (KU_*-valued) $\hat{A}$-genus (Todd genus), $\mathrm{MSpin}_* \to \mathrm{KO}_*$ ($\mathrm{MSpin}^c_* \to \mathrm{KU}_*$).

★ [176] 1964. Wall, *Graded Brauer groups.*
Shows that the set of equivalence classes of $\mathbb{Z}/2$-graded central simple algebras over a field k forms a group, called the *($\mathbb{Z}/2$-) graded Brauer group* (or *Brauer–Wall group*) of k, and shows that the $\mathbb{Z}/2$-graded Brauer group of $\mathbb{R}$ is $\mathbb{Z}/8$ and that of $\mathbb{C}$ is $\mathbb{Z}/2$, in both cases represented by the corresponding type of Clifford algebras under tensor product. The results are obtained for fields k more general than only $\mathbb{R}$ and $\mathbb{C}$.

★ [5] 1966. Adams, *On the groups J(X)–IV.*
Identifies a family of nontrivial elements in the stable homotopy groups of spheres, $\pi_* \mathbb{S}$, using KO-theory.

★ [10] 1966. Anderson–Brown–Peterson, *Spin cobordism.*
An announcement of [9] . Describes the additive structure and some of the multiplicative structure of the spin cobordism ring, MSpin_*, using KO-theory.

[5] a Clifford module is equivalent to zero if its module structure can be extended over a larger Clifford algebra.

★ [8] 1966. Anderson–Brown–Peterson, *SU-cobordism, KO-characteristic numbers, and the Kervaire invariant.*
Introduces KO-valued characteristic classes and uses them to study the SU-cobordism ring, MSU_*. Specifically, for each $i \in \mathbb{Z}_{\geq 0}$, the i-th KO-Pontrjagin class $\pi^i \in \mathrm{KO}^0(\mathrm{BSO}(2m))$ is defined as follows. Let $\mathrm{U}(1)^m \hookrightarrow \mathrm{SO}(2m)$ be the maximal torus, let X_j be the complex line bundle on $\mathrm{BU}(1)^m$ classified by the projection to the j-th factor of $\mathrm{BU}(1)$, and set $x_j = [X_j] - 1 \in \mathrm{KU}^0(\mathrm{BU}(1)^m)$. Define

$$\pi_t = \prod_{j=1}^{m}(1 + t(x_j + \bar{x}_j)) \in \mathrm{KU}^0(\mathrm{BU}(1)^m)[t],$$

where $\bar{x}_j = \frac{-x_j}{1+x_j}$ denotes the complex conjugate, and let $\tilde{\pi}^i$ be the $\mathrm{KU}^0(\mathrm{BU}(1)^m)$-valued coefficient of t^i in π_t. Then $\tilde{\pi}^i$ lifts to a unique element $\pi^i \in \mathrm{KO}^0(\mathrm{BSO}(2m))$ called the *i-th* KO-*Pontrjagin class*. Results include:

- An explicit comparison between the KO-Pontrjagin classes and the ordinary Pontrjagin classes of a vector bundle via the Pontrjagin (a.k.a. Chern) character map $\mathrm{KO}^0(X) \to \mathrm{H}^*(X;\mathbb{Q})$.
- Identifies when a class $[M] \in \mathrm{MSU}_*$ is zero in terms of the Chern numbers and KO-characteristic numbers of M.
- Identifies which classes in MSU_* contain framed manifolds, in other words, identifies the image of the unit map, $u : \mathbb{S} \to \mathrm{MSU}$, on homotopy groups. More specifically,

$$\mathrm{Im}(u_k) = \begin{cases} \mathbb{Z}/2, & k = 1,2 \\ 0, & k \neq 1,2 \end{cases} \mod 8.$$

- Contains another proof that the Kervaire invariant of an $(8r+2)$-dimensional framed manifold is zero.

The KO-characteristic classes introduced here are also the source of the splitting of MSpin found in [9] .

★ [70] 1966. Conner–Floyd, *The relation of cobordism to K-theories.*
Establishes the "Conner–Floyd isomorphisms",

$$\mathrm{MU}^*(X) \otimes_{\mathrm{MU}^*} \mathrm{KU}^* \cong \mathrm{KU}^*(X), \quad \mathrm{MSp}^*(X) \otimes_{\mathrm{MSp}^*} \mathrm{KO}^* \cong \mathrm{KO}^*(X);$$

$$\text{or } \mathrm{MU}_*(X) \otimes_{\mathrm{MU}_*} \mathrm{KU}_* \cong \mathrm{KU}_*(X), \quad \mathrm{MSp}_*(X) \otimes_{\mathrm{MSp}_*} \mathrm{KO}_* \cong \mathrm{KO}_*(X).$$

In other words, "complex cobordism determines complex K-theory" and "symplectic cobordism determines KO-theory".

★ [9] 1967. Anderson–Brown–Peterson, *The structure of the spin cobordism ring.* Contains the results that were announced in [10] . Describes the structure of the spin cobordism ring, MSpin_*, in terms of connective covers of KO-theory and suspensions of $\mathrm{H}\mathbb{Z}/2$. More specifically, let $\pi^i : \mathrm{BSpin} \to \mathrm{KO}$ be the pullback to BSpin of the i-th KO-Pontrjagin class (with the same name) defined in [8] , and let $\pi^I = \pi^{i_1} \dots \pi^{i_k}$, for $I = (i_1, ..., i_k)$. Results include:

- The map $\pi^I : \mathrm{BSpin} \to \mathrm{KO}$ factors as $\pi^I : \mathrm{BSpin} \to \mathrm{ko}\langle n_I \rangle \to \mathrm{KO}$, where
$$n_I = \begin{cases} 4|I|, & |I| \text{ is even} \\ 4|I| - 2, & |I| \text{ is odd,} \end{cases}$$
where $|I| = i_1 + \cdots + i_k$.

- Let $f_I : \mathrm{MSpin} \to \mathrm{ko}\langle n_I \rangle$ be the map that corresponds to π^I under the Thom isomorphism [26] . There exists a collection of elements $Z \subset \mathrm{H}^*(\mathrm{MSpin}; \mathbb{Z}/2)$, such that the map,
$$F_{\mathrm{ABP}} : \mathrm{MSpin} \xrightarrow{\left(\prod f_I\right) \times \left(\prod z\right)} \prod_{\substack{|I| \text{ even} \\ 1 \notin I}} \mathrm{ko}\langle 4|I| \rangle \times \prod_{\substack{|I| \text{ odd} \\ 1 \notin I}} \mathrm{ko}\langle 4|I| - 2 \rangle \times \prod_{z \in Z} \Sigma^{|z|}\, \mathrm{H}\,\mathbb{Z}/2,$$
induces an isomorphism in cohomology with $\mathbb{Z}/2$-coefficients. Since both the domain and the codomain of F_{ABP} are connective with (degree-wise) finitely generated homotopy groups, F_{ABP} is a 2-local equivalence.

- A(n unoriented) manifold M is (unoriented) cobordant to a spin manifold if and only if all Stiefel–Whitney numbers of M involving w_1 or w_2 vanish.

While the complex case is not discussed in this paper, the same methods and constructions apply to produce a 2-local splitting of MSpin^c (see [172] for details). Namely, let π^I also denote the composite $\mathrm{BSpin}^c \to \mathrm{BSO} \xrightarrow{\pi^I} \mathrm{KO} \xrightarrow{\otimes\mathbb{C}} \mathrm{KU}$.

- The map π^I factors as $\pi^I : \mathrm{BSpin}^c \to \mathrm{ku}\langle 4|I| \rangle \to \mathrm{KU}$.

- Let $f_I : \mathrm{MSpin}^c \to \mathrm{ku}\langle 4|I| \rangle$ be the map that corresponds to π^I under the Thom isomorphism [26] . There exists a collection of elements $Z \subset \mathrm{H}^*(\mathrm{MSpin}^c; \mathbb{Z}/2)$, such that the map,
$$F^c_{\mathrm{ABP}} : \mathrm{MSpin}^c \xrightarrow{\left(\prod f_I\right) \times \left(\prod z\right)} \prod_I \mathrm{ku}\langle 4|I| \rangle \times \prod_{z \in Z} \Sigma^{|z|}\, \mathrm{H}\,\mathbb{Z}/2,$$
is a 2-local equivalence.

★ [12] 1968. Anderson–Hodgkin, *The K-theory of Eilenberg–MacLane complexes.*
Computes the K-theory of Eilenberg–Mac Lane spaces, $\mathrm{KU}^*(K(\pi, n))$. Also includes an appendix which contains results for $\mathrm{KO}^*(K(\pi, n))$. Write the actual results in both cases.

★ [25] 1968. Atiyah, *Bott periodicity and the index of elliptic operators.*
Uses index theory to prove generalizations of the 8-fold Bott periodicity of KO, the 2-fold Bott periodicity of KU, and the $(1,1)$-fold Bott periodicity of KR in the equivariant settings. The most general statement from which all others follow is the following.

- Let $G \to \mathrm{Spin}^c(p, q)$ be a C_2-equivariant homomorphism of groups with C_2-action, and suppose $p \equiv q \mod 8$. Then there is a class $u \in \mathrm{KR}_G(\mathbb{R}^{p,q})$, such that multiplication by u induces an isomorphism,

  $$\mathrm{KR}_G(X) \cong \mathrm{KR}_G(\mathbb{R}^{p,q} \times X).$$

  Here, C_2 acts on $\mathbb{R}^{p,q} = \mathbb{R}^{p+q}$ trivially on the $\mathbb{R}^p$ factor and by -1 on the $\mathbb{R}^q$ factor, and $\mathrm{Spin}^c(p, q)$ is the spin group in the Real Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{R}^{p,q})$ with the induced C_2-action.

The corresponding Thom isomorphisms are formal consequences of the equivariant periodicity theorems.

★ [118] 1968. Karoubi, *Algèbres de Clifford et K-théorie.*
Explores further the relationship between Clifford algebras and K-theory introduced in [26] . Includes both the real and complex cases. need to add more details here.

★ [172] 1968. Stong, *Notes on cobordism theory.*
An exposition of many topics in cobordism theory, including detailed treatments of various relationships between spin and spin^c bordism with topological K-theory (Chapter XI). The following link contains a detailed table of contents which was compiled and made available by Landweber and Ravenel. Contents include:

- a computation of $\mathrm{H}^*(\mathrm{BSpin}; \mathbb{Z}/2)$ (p. 291-292) and $\mathrm{H}^*(\mathrm{BSpin}^c; \mathbb{Z}/2)$ (p. 293);
- if a vector bundle admits a KO-orientation, then it must admit a spin structure (p. 301);
- an exposition of the main results of Anderson–Brown–Peterson [9] (p. 302-336);
- all torsion in $\mathrm{H}_*(\mathrm{BSpin}; \mathbb{Z})$ and $\mathrm{H}_*(\mathrm{BSpin}; \mathbb{Z})$ is of order 2 (p. 316-317);
- two spin manifolds are spin cobordant if and only if they have the same KO-characteristic numbers and the same Stiefel–Whitney numbers (p. 337);
- two spin^c manifolds are spin^c cobordant if and only if they have the same rational characteristic numbers and the same Stiefel–Whitney numbers (p. 337);

- the ring $(\mathrm{MSpin}^c_*/\mathrm{torsion}) \otimes \mathbb{Z}/2$ is a polynomial algebra over $\mathbb{Z}/2$ on generators y_i, with $|y_i| \in \{2, 4k \mid k \in \mathbb{Z}_{\geq 1}\}$ (p. 347);

- after taking homotopy groups, the forgetful map $\mathrm{MU} \to \mathrm{MSpin}^c$ induces a surjection $\mathrm{MU}_* \to \mathrm{MSpin}^c_*/\mathrm{torsion}$ (p. 348);

- if $\eta : \mathbb{S} \to \mathrm{MSpin}$ is the forgetful map (i.e. the unit), then on homotopy groups, $\pi_0\eta$ is an isomorphism, $\pi_n\eta = 0$ for $n \not\equiv 1, 2 \mod 8$, and $\mathrm{Im}(\pi_n\eta) = \mathbb{Z}/2$, for $n \equiv 1, 2 \mod 8$ (p. 350);

- if $\eta^c : \mathbb{S} \to \mathrm{MSpin}^c$ is the forgetful map (i.e. the unit), then on homotopy groups, $\pi_0\eta^c$ is an isomorphism, $\pi_n\eta^c = 0$ for $n > 0$ (p. 350).

★ [33] 1969. Atiyah–Segal, *Equivariant K-theory and completion.*
Proves the "Atiyah–Segal completion theorem" for all compact Lie groups. The theorem concerns the relationship between the equivariant K-theory, $\mathrm{K}^*_G(X)$, of a G-space, X, and the non-equivariant K-theory of its homotopy quotient (a.k.a. Borel construction), $\mathrm{K}^*(X_G) = \mathrm{K}^*(X \times_G \mathrm{E}G)$. In other words, this is the question of whether or not the G-spectrum K_G is "Borel complete", and identifying the corresponding completion map. Results include:

- **Atiyah–Segal completion theorem:** Let G be a compact Lie group and X be a compact G-space. Assume that $\mathrm{KU}^*_G(X)$ is finite over the representation ring $\mathrm{R}(G) = \mathrm{KU}^0_G(*)$. Let I_G be the augmentation ideal in $\mathrm{R}(G)$, and let

$$\mathrm{KU}^*_G(X)^\wedge_{I_G} := \lim_n \Big(\mathrm{KU}^*_G(X)/(I^n_G \cdot \mathrm{KU}^*_G(X)) \Big).$$

  Then the associated bundle construction induces an isomorphism,

$$\mathrm{KU}^*_G(X)^\wedge_{I_G} \cong \mathrm{KU}^*(X_G).$$

- Specializing to the case $X = *$ yields

$$\mathrm{R}(G)^\wedge_{I_G} \cong \mathrm{KU}^0(\mathrm{B}G).$$

- Proves an analogous result for KR-theory, which has implications for KO-theory as well, though they note that the proof does not work if one works entirely with KO-theory. Recovers the computations of [7] .

I don't know whether the finiteness assumption is necessary or not. In [33] , they indicate situations in which it is satisfied, but I often see the condition omitted in other references. In the case $X = *$ and G finite, the completion theorem was proven in [22] , and for $X = *$ and G connected, it was proven in [28] . Note that these statements and proofs did not involve equivariant K-theory.

★ [37] 1969. Atiyah–Singer, *Index theory for skew-adjoint Fredholm operators.*
Constructs Ω-spectra representing KU and KO out of skew-adjoint Clifford linear Fredholm operators. Includes a proof of Bott periodicity for the spaces in these spectra. The results in this paper can be seen as a generalization of the Atiyah–Jänich theorem (Appendix of [40] , and [114]) from degree 0 K-theory to arbitrary degree.

★ [79] 1970. Donovan–Karoubi, *Graded Brauer groups and K-theory with local coefficients.*
Defines twisted K-theory groups, $\mathrm{KU}^{\alpha}(X)$ and $\mathrm{KO}^{\beta}(X)$, for twists,

$$\alpha \in \mathbb{Z}/2 \times \mathrm{H}^1(X;\mathbb{Z}/2) \times \mathrm{H}^3(X;\mathbb{Z})_{\mathrm{tors}},$$

and

$$\beta \in \mathbb{Z}/8 \times \mathrm{H}^1(X;\mathbb{Z}/2) \times \mathrm{H}^2(X;\mathbb{Z}/2),$$

where A_{tors} denotes the torsion subgroup of an abelian group A. (What is the space $K(A,n)_{\mathrm{tors}}$ that classifies $\mathrm{H}^n(-;A)_{\mathrm{tors}}$? Or does it even exist?) The groups classifying the twists α and β over a space X arise from considering the $\mathbb{Z}/2$-graded Brauer groups of bundles of (real or complex) $\mathbb{Z}/2$-graded central simple algebras on X. When X is a point, these groups are given by the Morita classes of Clifford algebras ([26] , [176]). Compare the groups of twists above to the results of [133] ; namely, the twists for K-theory considered here are some, but not all, of the possible twists of K-theory. Generalizes the Thom isomorphism in K-theory ([40] , [26]) to all real vector bundles, where the degree shift must now include non-integral twists. This was the initial motivation for considering the types of twists that they did. Also includes a discussion of the multiplicative structure and Adams operations [1] in twisted K-theory. See [50] for results on the relationship between power operations and Thom classes in the general context of twisted equivariant KR-theory of topological stacks.

★ [39] 1971. Atiyah–Singer, *The index of elliptic operators: V.*
Proves a families index theorem for real elliptic operators. This makes essential use of KR-theory.

★ [166] 1971. Segal, *Equivariant stable homotopy theory.*
Introduces $\mathrm{RO}(G)$-graded equivariant stable homotopy theory for a finite group G, as well as an equivariant J-homomorphism, $J : \mathrm{KO}_G^{-1}(X) \to \mathrm{gl}_1\, \mathbb{S}_G^0(X)$. Identifies the equivariant stable cohomotopy ring of a point (in degree 0) as the Burside ring, $\mathrm{A}(G)$, of the group G:

$$\mathbb{S}_G^0(*) = \pi_0\, \mathbb{S}_G := \operatorname*{colim}_V [S^V, S^V]_G \cong \mathrm{A}(G),$$

where V is a real G-representation, and $\mathrm{A}(G)$ is the Grothendieck group of the category of finite G-sets.

★ [47] 1974. Becker, *Characteristic classes and K-theory.*
Proves an analogue of the results of [164] in the KO case. Results include:

- The inclusion $\mathrm{BO}(2) \to \mathrm{BO}$ induces a split surjection,
$$(\Sigma^{\infty}_{+} \mathrm{BO}(2))^0(X) \to \mathrm{KO}^0(X).$$

See [150] for an analogous result in the KR case.

★ [174] 1974. Thomas, *A relation between K-theory and cohomology.*
Establishes a refinement of the rational isomorphism $\mathrm{KU}^*(X) \otimes \mathbb{Q} \cong \mathrm{H}^{\mathrm{even}}(X; \mathbb{Q})$ of [28] , which includes some torsion, using the theory of λ-rings. Results include:

- For $n \in \mathbb{Z}_{>0}$, let l_n be the set of primes less than n. If X is a CW-complex of dimension at most $4n+3$ and $\mathrm{H}^{4*}(X; \mathbb{Z})$ has no l_{2n+1}-torsion, then there is an isomorphism of abelian groups $\mathrm{KO}(X) \cong \mathrm{H}^{4*}(X; \mathbb{Z})$ modulo 2-torsion.

★ [133] 1977. Madsen–Snaith–Tornehave, *Infinite loop maps in geometric topology.*
Shows that there are equivalences of infinite loop spaces,
$$\mathrm{GL}_1 \mathrm{KU} \simeq K(\mathbb{Z}/2, 0) \times K(\mathbb{Z}, 2) \times \mathrm{BSU}_{\otimes},$$
and
$$\mathrm{GL}_1 \mathrm{KO} \simeq K(\mathbb{Z}/2, 0) \times K(\mathbb{Z}/2, 1) \times \mathrm{BSO}_{\otimes}.$$
These results classify (one connected component of) twists of K-theory by the spaces
$$\mathrm{BGL}_1 \mathrm{KU} \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times \mathrm{BBSU}_{\otimes},$$
and
$$\mathrm{BGL}_1 \mathrm{KO} \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2) \times \mathrm{BBSO}_{\otimes}.$$

★ [119] 1978. Karoubi, *K-theory: An introduction.*
A book for learning topological K-theory using the language of Clifford algebras, which handles both the real and complex cases explicitly. Defines K-theory in the setting of Banach categories.

★ [141] 1979. Meier, *Complex and real K-theory and localization.*
Proves that localization with respect to KU and KO coincide (see also [159] and [58] for further analysis of K-local spectra, i.e. spectra that are local with respect to either KU or KO). More specifically, shows that for any CW complex, X, and any abelian group A, $\widetilde{\mathrm{KU}}_*(X; A) = 0$ if and only if $\widetilde{\mathrm{KO}}_*(X; A) = 0$. Applies this result to conclude that the spaces in the KO-theory Ω-spectrum are KU-local, and vice versa. Similarly, Spin as well as any homogeneous space obtained from the stable classical groups $\mathrm{O}, \mathrm{SO}, \mathrm{U}, \mathrm{SU}, \mathrm{Sp}$, and their loop spaces, are K-local. Computes the homotopy groups of the K-localization of $\mathrm{BU}[2n]$ (the $(2n-1)$-connective cover of BU) for $n \geq 2$.

★ [122] 1981. Kasparov, *The operator K-functor and extensions of C^*-algebras.*
Introduces bivariant KK-theory for C^*-algebras, along with its product, which gives KK the structure of an additive category. Works in a general setting which applies to the real, complex, and Real cases, as well as the equivariant setting, for G-C^*-algebras. Proves a version of Bott periodicity for KK which is as general as the one in [25] when applied to commutative C^*-algebras.

★ [159] 1984. Ravenel, *Localization with respect to certain periodic homology theories.*
Shows that the Bousfield classes of KO and KU are the same, i.e. they induce the same localization functor on the category of spectra (see [141] for an earlier proof, and [58] for an analysis of the K-local category). Proves the "Smash product theorem", i.e. for $p > 2$ and any spectrum X, we have $L_1 X \simeq X \wedge L_1 \mathbb{S}$, where L_1 denotes localization with respect to p-local K-theory.

★ [123] 1988. Kasparov, *Equivariant KK-theory and the Novikov conjecture.*
A follow-up to Kasparov's work on KK-theory [122] . Applies KK-theory to prove the Novikov conjecture in a certain class of cases. more details. The results first appeared in a 1981 preprint which has since been published as [124] in a modified form.

★ [132] 1989. Lawson–Michelsohn, *Spin geometry.*
A book that introduces the basics of spin geometry, including relations with topological K-theory and applications to topology and geometry. Topics include: a detailed introduction to Clifford algebras and their modules, including the Atiyah–Bott–Shapiro construction, which is extended to the KR case using "Real" Clifford modules; a brief introduction to KU, KO, and KR; spinor bundles and Dirac operators; Thom isomorphisms and the Chern character; various versions of the Atiyah–Singer index theorem; and many applications to topology and differential geometry, including metrics of positive scalar curvature and group actions on manifolds. Also contains useful appendices on principal bundles, classifying spaces, characteristic classes, and Thom classes.

★ [144] 1989. Miller, *Vector fields on spheres, etc.*
(See [1] .) Notes from a course taught by Haynes Miller at MIT in the fall of 1989 in honor of Frank Adams. They were originally taken by Matthew Ando, and were later transcribed into LaTeX by Eric Peterson, with figures by Yi-Wei Chan, Meng Guo, and Matt Mahowald. The notes were further edited by Michael Donovan and Aaron Mazel-Gee.

★ [58] 1990. Bousfield. *A classification of K-local spectra.*
Classifies spectra which are K-local, i.e. local with respect to either KU or KO (recall that KU and KO have the same Bousfield class [141] , [159]).

★ [151] 1991. Ochanine, *Elliptic genera, modular forms over* KO_**, and the Brown–Kervaire*

invariant.
Shows that, in contrast with [70] ,

$$\mathrm{MSU}^*(X) \otimes_{\mathrm{MSU}^*} \mathrm{KO}^* \ncong \mathrm{KO}^*(X).$$

need more here. also need more on what in this paper implies the above result.

★ [111] 1992. Hopkins–Hovey, *Spin cobordism determines real K-theory.*
Proves analogues of the “Conner–Floyd isomorphisms” [70] for spin and spin^c bordism, i.e.

$$(\mathrm{MSpin}^c)^*(X) \otimes_{(\mathrm{MSpin}^c)^*} \mathrm{KU}^* \cong \mathrm{KU}^*(X), \quad \mathrm{MSpin}^*(X) \otimes_{\mathrm{MSpin}^*} \mathrm{KO}^* \cong \mathrm{KO}^*(X);$$

$$\text{or } \mathrm{MSpin}^c_*(X) \otimes_{\mathrm{MSpin}^c_*} \mathrm{KU}_* \cong \mathrm{KU}_*(X), \quad \mathrm{MSpin}_*(X) \otimes_{\mathrm{MSpin}_*} \mathrm{KO}_* \cong \mathrm{KO}_*(X).$$

Note that the proof in this paper does not use a geometric model for K-theory whose (co)cycles are spin(^{c}) manifolds to establish these isomorphisms. Rather, the proof proceeds algebraically by localizing at each prime.

★ [126] 1993. Kreck–Stolz, *$\mathbb{HP}^2$-bundles and elliptic homology.*
Provides a geometric model of KO-homology as a quotient of spin bordism via the Atiyah–Bott–Shapiro map $\mathrm{MSpin} \to \mathrm{KO}$ [26] . In particular, the kernel of this map is identified explicitly in geometric terms, namely, as the subgroup of MSpin_* consisting of bordism classes containing total spaces of $\mathbb{HP}^2$-bundles. Provides a similar geometric model of elliptic homology.

★ [85] 1995. Fajstrup, *Tate cohomology of periodic K-theory with reality is trivial.*
Shows that the Tate fixed points of KR vanish. This implies that the homotopy and genuine C_2-fixed points of KR are equivalent,

$$\mathrm{KR}^{hC_2} \simeq \mathrm{KR}^{C_2} \simeq \mathrm{KO}\,.$$

★ [124] 1995. Kasparov, *K-theory, group C^*-algebras, and higher signatures (Conspectus) (first distributed 1981).*
A (modified) preprint version of [123] .

★ [177] 1998. Wolbert, *Classifying modules over K-theory spectra.*
Let K denote either KU or KO. Provides an algebraic criterion for determining when a K-local spectrum is a K-module. Shows that any homotopy K-module is equivalent to K-module, and that any ko_*-module can be realized as the homotopy of a ko-module.

★ [115] 2001. Joachim, *A symmetric ring spectrum representing KO-theory.*

Proves that the ring spectrum KO carries the structure of an $\mathbb{E}_\infty$-ring spectrum by constructing a commutative symmetric ring spectrum (i.e. commutative monoid in the symmetric monoidal category of symmetric spectra [134]) which represents KO-theory via Clifford linear Fredholm operators, along the lines of the model of KO in [37] .

★ [131] 2003. Laures, *An E_∞ splitting of spin bordism.*
Establishes a multiplicative splitting,

$$\mathrm{MSpin} \simeq T_\zeta \wedge \bigwedge_{i=1}^{\infty} T\,\mathbb{S},$$

where ζ is the generator of $\pi_{-1} L_{K(1)}\,\mathbb{S}$, T_ζ is the $\mathbb{E}_\infty$-cone over ζ, and $T\,\mathbb{S}$ is the free $\mathbb{E}_\infty$-ring generated by the sphere, $\mathbb{S}$. Derives the following consequences:

- The spin bordism ring, MSpin_* is the free θ-algebra[6] over KO_* in (countably) infinitely many generators, and an algorithm is given for constructing the generators.
- Shows that, $K(1)$-locally, the Atiyah–Bott–Shapiro orientation [26] , MSpin $\to$ KO, refines to an $\mathbb{E}_\infty$-map (see also [116] and [16]).
- Recovers the Conner–Floyd [70] isomorphism of Hopkins–Hovey [111] .

In the process, also shows that the map

$$\mathrm{KO}^0(\mathrm{MSpin}) \to \mathrm{Hom}_{\mathrm{cts}}(\pi_0\,\mathrm{KO} \wedge \mathrm{MSpin}, \pi_0\,\mathrm{KO} \wedge \mathrm{KO}),$$

is injective, i.e. that a map MSpin $\to$ KO is determined by its induced map in KO-homology.

★ [42] 2005. Baker–Richter, *Invertible modules for commutative $\mathbb{S}$-algebras with residue fields.*
Provides general criteria for when the Picard group of a commutative ring spectrum, R, is isomorphic to the Picard group of its homotopy ring, R_*. Examples include $R \in \{\mathrm{KU}, \mathrm{KO}[\frac{1}{2}], \mathrm{ku}, \mathrm{ko}\}$. In particular, this leads to the fact that $\mathrm{Pic}(\mathrm{KU}) \cong \mathbb{Z}/2$. For a computation of Pic(KO), see [136] .

★ [43] 2005. Baker–Richter, *On the Γ-cohomology of rings of numerical polynomials and E_∞ structures on K-theory.*
Proves that there are unique $\mathbb{E}_\infty$-structures lifting the usual homotopy ring structures on KU and KO. For the connective versions, see [44] .

★ [44] 2008. Baker–Richter, *Uniqueness of E_∞ structures for connective covers.*

[6] A *θ-algebra* is a commutative algebra over a ring together with a unary operation that satisfies certain properties. The homotopy groups of a $K(1)$-local $\mathbb{E}_\infty$-ring carry the structure of a θ-algebra.

Proves that there are unique $\mathbb{E}_\infty$-structures lifting the usual homotopy ring structures on ku and ko. For the periodic versions, see [43] .

★ [125] 2009. Kitaev, *Periodic table for topological insulators and superconductors.*
Classifies gapped phases of noninteracting fermions in terms of KU-theory and KO-theory using a version of the Atiyah–Bott–Shapiro construction [26] relating Clifford algebras to K-theory as developed in the book [119] .

★ [16] 2010. Ando–Hopkins–Rezk, *Multiplicative orientations of KO-theory and of the spectrum of topological modular forms.*
Computes π_0 of the space of $\mathbb{E}_\infty$-maps from MSpin to KO (which is also equal to π_0 of the space of $\mathbb{E}_\infty$-maps from MString to KO). Results include:

- There is a bijection between $\pi_0\,\mathbb{E}_\infty(\mathrm{MSpin}, \mathrm{KO})$ and the set of even sequences $b : \mathbb{Z}_{\geq 4} \to \mathbb{Q}$, such that $b_k = -\dfrac{B_k}{2k} \mod \mathbb{Z}$, where B_k is the kth Bernoulli number, and for each prime p and each $c \in \mathbb{Z}_p^\times$, the sequence $\{(1-c^k)(1-p^{k-1})b_k\}_{k\geq 4}$ satisfies the "generalized Kummer congruences".
- The sequence in the above classification that corresponds to the Atiyah–Bott–Shapiro orientation ([26] , [116]) is $b_k = -\dfrac{B_k}{2k}$.

The proofs are purely homotopical, and uses the machinery developed in [15] .

★ [61] 2010. Bruner–Greenlees, *Connective real K-theory of finite groups.*
A book about equivariant connective real K-theory, ko_G and kr_G, which contains both a significant amount of exposition, as well as many computations. Includes:

- the $\mathrm{RO}(C_2)$-graded homotopy groups of kr;
- an equivalence $(\Sigma^{-\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma\,\mathrm{ko}$;
- an equivalence $(\Sigma^{-2\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma^{-6}\,\mathrm{ko}\langle 4\rangle$;
- an equivalence $(\Sigma^{-3\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma^{-5}\,\mathrm{ko}\langle 2\rangle$;
- a cofiber sequence $\mathrm{ko} \to \mathrm{kr}^{hC_2} \to \bigvee_{k\geq 1} \Sigma^{-4k}\,\mathrm{H}\,\mathbb{Z}/2$;
- a cofiber sequence $\mathrm{kr}_{hC_2} \to \mathrm{ko} \to \bigvee_{k\geq 0} \Sigma^{4k}\,\mathrm{H}\,\mathbb{Z}/2$;
- the ko homology, $\mathrm{ko}_*(\mathrm{B}G)$, and cohomology, $\mathrm{ko}^*(\mathrm{B}G)$, of the classifying spaces of various groups, including cyclic groups, generalized quaternion groups, the Klein 4-group, dihedral 2-groups, and elementary abelian 2-groups;

★ [66] 2010. Bunke–Schick, *Uniqueness of smooth extensions of generalized cohomology theories.*
Introduces axioms for differential refinements (a.k.a. smooth extensions), $\widehat{E}$, of cohomology theories, E, and proves a corresponding uniqueness theorem for such refinements under certain extra conditions (including the presence of additional structure, such as "integration" or a multiplication). In particular, the differential cohomology theories constructed in [112] satisfy the axioms, and KU and KO satisfy the additional conditions of the uniqueness theorem, implying that any two multiplicative differential refinements of $\widehat{\mathrm{KU}}$ or $\widehat{\mathrm{KO}}$ that satisfy the axioms are isomorphic. They also give examples of non-isomorphic differential refinements of KU-theory in the absence of a multiplicative structure or integration.

★ [110] 2010. Hohnhold–Stolz–Teichner, *From minimal geodesics to super symmetric field theories.*
Describes equivalences between eight models for the spaces in (an Ω-spectrum model for) the KO-theory spectrum, including new ones related to supersymmetric Euclidean field theories. These spaces can be roughly describes as follows.

1. Bott's spaces of minimal geodesics [56] .
2. Milnor's spaces of Clifford module structures [145] .
3. Spaces of Clifford-linear unbounded operators.
4. Configuration spaces of finitely many points in a circle labeled by finite dimensional Clifford modules.
5. Spaces of super semigroups of finite rank Clifford-linear operators.
6. Spaces of supersymmetric positive Euclidean field theories of dimension $1|1$.
7. The classifying spaces of certain Quillen-style categories that "categorify" the Atiyah–Bott–Shapiro description of KO_* [26] .
8. Atiyah–Singer's spaces of Clifford-linear Fredholm operators [37] .

Also describes a symmetric ring spectrum representing KO-theory similar to the one in [115] , but using unbounded operators (3) instead of Fredholm operators (8).

★ [18] 2014. Antieau–Gepner–Gómez, *Actions of $K(\pi, n)$ spaces on K-theory and uniqueness of twisted K-theory.*
Proves that

- $[K(\mathbb{Z},3), \mathrm{BGL}_1\,\mathrm{KU}] \cong [K(\mathbb{Z},3), K(\mathbb{Z},3)] \cong \mathbb{Z}$, and
- $[K(\mathbb{Z}/2,2), \mathrm{BGL}_1\,\mathrm{KO}] \cong [K(\mathbb{Z}/2,2), K(\mathbb{Z}/2,2)] \cong \mathbb{Z}/2$.

This allows one to compare various geometric definitions of twisted KU-theory (resp. KO-theory) twisted by elements of $\mathrm{H}^3(-;\mathbb{Z})$ (resp. $\mathrm{H}^2(-;\mathbb{Z}/2)$) via the homotopy class of the corresponding map $K(\mathbb{Z},3) \to \mathrm{BGL}_1\,\mathrm{KU}$ (resp. $K(\mathbb{Z}/2,2) \to \mathrm{BGL}_1\,\mathrm{KO}$). In particular, in the real case, all nontrivial ways of defining twisted KO-theory for twists classified by $K(\mathbb{Z}/2,2)$ are equivalent. Also contains an appendix which explains how to adapt the geometric model for KO-theory given in [115] to the twisted case, within the general framework for twisted cohomology set up in [15] , [14] , and [17] .

★ [106] 2014. Heard–Stojanoska, *K-theory, reality, and duality.*
Obtains duality results for KO-theory by interpreting it as both the homotopy fixed points and the homotopy orbits of KR. Also includes a discussion of the differentials in the homotopy fixed point spectral sequence for $\mathrm{KR}^{C_2} \simeq \mathrm{KO}$. Results include:

- There is an equivalence, $\mathrm{KR}_{hC_2} = \mathrm{KU}_{hC_2} \simeq \mathrm{KO}$, between C_2-homotopy orbits of KR, i.e. the homotopy orbits of the complex conjugation action on KU, and KO.
- The Anderson dual of KR is equivalent to $\Sigma^4\,\mathrm{KR}$, which implies that the Anderson dual of KO is equivalent to $\Sigma^4\,\mathrm{KO} \simeq \mathrm{KSp}$.
- The $K(1)$-local Brown-Comenetz dual of KO is equivalent to $\Sigma^5\,\mathrm{KO}$.

★ [136] 2016. Mathew–Stojanoska, *The Picard group of topological modular forms via descent theory.*
Computes the Picard group $\mathrm{Pic}(\mathrm{KO}) \cong \mathbb{Z}/8$ using C_2-Galois descent for the C_2-Galois extension $\mathrm{KO} \to \mathrm{KU}$, together with the fact that $\mathrm{Pic}(\mathrm{KU}) \cong \mathrm{Pic}(\mathrm{KU}_*) \cong \mathbb{Z}/2$ (see [42]). The computation uses the homotopy fixed point spectral sequence for $\mathfrak{pic}(\mathrm{KU})$ with C_2-action induced from complex conjugation on KU, where $\mathfrak{pic}$ denotes the Picard spectrum.

★ [161] 2016. Rosenberg, *Structure and applications of real C^*-algebras.*
An expository survey of real C^*-algebras, including their K-theory. Emphasizes aspects of the theory and applications that are different for the real case. Topics include metrics of positive scalar curvature, orientifold string theory, and the Baum–Connes conjecture.

★ [105] ∼2017. Hatcher, *Vector bundles and K-theory.*
An unfinished introductory text on vector bundles, topological K-theory, and characteristic classes.

★ [142] 2017. Merling, *Equivariant algebraic K-theory of G-rings.*
Introduces a notion of algebraic K-theory for topological G-rings, $\mathrm{K}_G^{\mathrm{top}}$, and shows that

- $\mathrm{K}_G^{\mathrm{top}}(\mathbb{R}, \text{trivial action}) \simeq \mathrm{ko}_G$,
- $\mathrm{K}_G^{\mathrm{top}}(\mathbb{C}, \text{trivial action}) \simeq \mathrm{ku}_G$,

- $\mathrm{K}^{\mathrm{top}}_{C_2}(\mathbb{C}, \text{complex conjugation}) \simeq \mathrm{kr}$,

where the G-rings $\mathbb{R}$ and $\mathbb{C}$ are considered with their topology. Does the same method work to show $\mathrm{K}^{\mathrm{alg}}_{G\rtimes C_2}(\mathbb{C}, \text{trivial} \rtimes \text{complex conjugation}) \simeq \mathrm{kr}_G$, for any Real group G?

★ [178] 2017. Zhang, *A mod 2 index theorem for pin$^-$ manifolds.*
Extends the mod 2 index theorem of Atiyah and Singer [39] to real vector bundles over $(8k+2)$-dimensional compact pin$^-$ manifolds. The analytic index is the reduced η invariant of twisted Dirac operators and the topological index is defined in terms of KO-theory.

★ [128] 2018. Land–Nikolaus, *On the relation between K- and L-theory of C^*-algebras.*
Studies comparison maps between topological K-theory and algebraic L-theory of C^*-algebras. Results include:

- There is a bijection
$$\mathrm{ev}_{\mathbb{C}} : \pi_0 \operatorname{Map}_{\mathrm{Fun}(C^*\mathrm{Alg},\mathrm{Sp})}(\mathrm{ku}, \ell) \to \mathbb{Z}\,.$$
There exists $\tau : \mathrm{ku} \to \ell$ that can be made lax symmetric monoidal, and all such τ have $\mathrm{ev}_{\mathbb{C}}(\tau) = 1$.

- There are equivalences of lax symmetric monoidal functors,
$$\mathrm{KU}[\tfrac{1}{2}] \simeq \mathrm{L}[\tfrac{1}{2}] \quad \text{and} \quad \mathrm{KO}[\tfrac{1}{2}] \simeq \mathrm{L}[\tfrac{1}{2}],$$
where in the first equivalence, the functors are on the category of complex C^*-algebras, and in second equivalence, the functors are on the category of real C^*-algebras.

- In the category of spectra, $[\mathrm{KU}, \mathrm{L}(\mathbb{C})] = [\mathrm{L}(\mathbb{C}), \mathrm{KU}] = [\ell(\mathbb{C}), \mathrm{ku}] = 0$.

★ [147] 2019. Mohsen, *Chern–Simons invariants in KK-theory.*
Realizes the Chern–Simons invariants ([68] , [29] , [30] , [31]) as elements of real KK-theory.

★ [103] 2020. Guillou–Hill–Isaksen–Ravenel, *The cohomology of C_2-equivariant $\mathcal{A}(1)$ and the homotopy of ko_{C_2}.*
Computes the C_2-equivariant mod 2 cohomology of ko_{C_2}, the C_2-equivariant connective cover of KO_{C_2}, and uses it to compute the $RO(C_2)$-graded homotopy ring of ko_{C_2}. Results include:

- The C_2-fixed point spectrum of ko_{C_2} is $(\mathrm{ko}_{C_2})^{C_2} \simeq \mathrm{ko} \vee \mathrm{ko}$.

- Let σ be the 1-dimensional sign C_2-representation. The C_2-fixed point spectrum of $\Sigma^\sigma \mathrm{ko}_{C_2}$ is $(\Sigma^\sigma \mathrm{ko}_{C_2}) \simeq \mathrm{ko}$.

- The Hopf map induces a C_2-equivariant map $\eta : \Sigma^\sigma \mathrm{ko}_{C_2} \to \mathrm{ko}_{C_2}$, which on C_2-fixed points, induces the map $\eta^{C_2} = (-1, 1) : \mathrm{ko} \to \mathrm{ko} \vee \mathrm{ko}$.

- Complexification $\mathrm{KO}_{C_2} \to \mathrm{KR}$ induces a map on connective covers $c : \mathrm{ko}_{C_2} \to \mathrm{kr}$, which participates in a cofiber sequence,

$$\Sigma^\sigma \mathrm{ko}_{C_2} \xrightarrow{\eta} \mathrm{ko}_{C_2} \xrightarrow{c} \mathrm{kr}\,.$$

  In terms of spectral Mackey functors [cite, also discuss the spectral Mackey functor form of KR somewhere], this cofiber sequence can be written as

$$\begin{tikzcd}
\mathrm{ko} \arrow[r, "(-1{,}1)"] \arrow[d, bend right, "0"'] & \mathrm{ko} \vee \mathrm{ko} \arrow[r, "\nabla"] \arrow[d, bend right, "\nabla"'] & \mathrm{ko} \arrow[d, bend right, "c"'] \\
\Sigma^1 \mathrm{ko} \arrow[r, "\eta"'] \arrow[u, bend right, "\eta"'] \arrow[loop below, "\text{sign}"] & \mathrm{ko} \arrow[r, "c"'] \arrow[u, bend right, "\Delta"'] \arrow[loop below, "\text{trivial}"] & \mathrm{ku}, \arrow[u, bend right, "r"'] \arrow[loop below, "\text{conj}"]
\end{tikzcd}$$

  where $r : \mathrm{ku} \to \mathrm{ko}$ is induced by taking the real vector bundle underlying a complex vector bundle.

- The C_2-equivariant cohomology of ko_{C_2} with coefficients in the constant Mackey functor $\underline{\mathbb{Z}/2}$ is

$$\mathrm{H}^\star(\mathrm{ko}_{C_2}; \underline{\mathbb{Z}/2}) \cong \mathcal{A}^{C_2} /\!/ \mathcal{A}^{C_2}(1),$$

  where $\mathcal{A}^{C_2}$ is the C_2-equivariant mod 2 Steenrod algebra.

- The geometric fixed points of ko_{C_2} is $(\mathrm{ko}_{C_2})^{gC_2} \simeq \bigvee_{k\geq 0} \Sigma^{4k}\mathrm{H}\hat{\mathbb{Z}}_2$, where $\hat{\mathbb{Z}}_2$ is the 2-adic integers.

★ [45] 2022. Balderrama, *K-theory equivariant with respect to an elementary abelian 2-group.*
Computes the $RO(G)$-graded homotopy Mackey functor of KU_G and KO_G for $G = (\mathbb{Z}/2)^n$, together with product structures and power operations.

★ [169] 2022. Stolz, *Positive scalar curvature – constructions and obstructions.*
An expository survey on the state of the question:

> "Which closed connected smooth manifolds of dimension $n \geq 5$ admit a Riemannian metric whose scalar curvature is everywhere positive?"

as of 2022. The question has been resolved for simply connected manifolds, as well as for various other values of the fundamental group. The particular form of the classification, and the class of fundamental groups for which it has been solved, differs for spin and non-spin manifolds. The following are a combination of many prior results by different people.

- Let M be a non-spin manifold whose universal cover is also non-spin. If the fundamental group of M is one of: $e, (\mathbb{Z}/2)^r, (\mathbb{Z}/p)^r$ for p an odd prime and $r < \dim(M)$, then M admits a positive scalar curvature metric.

- Let M be a spin manifold with fundamental group π, let $f : M \to \mathrm{B}\pi$ be the map classifying the universal cover of M, and let $\alpha_\pi : \mathrm{MSpin}_*(\mathrm{B}\pi) \to \mathrm{KO}_*(\mathrm{B}\pi) \to \mathrm{KO}_*(C^*\pi)$ be the composite of the Atiyah–Bott–Shapiro map [26] on $\mathrm{B}\pi$ and the real assembly map. Then for $\pi = e, \mathbb{Z}/2, \mathbb{Z}/(2k+1), (\mathbb{Z}/p)^r$ for p an odd prime and $r > \dim(M)$, or if π has periodic cohomology, M admits a positive scalar curvature metric if and only if $\alpha_\pi([M, f]) = 0 \in \mathrm{KO}_n(C^*\pi)$.

★ [100] 2023. Gomi–Yamashita, *Differential KO-theory via gradations and mass terms.*
Refines the model of KO-theory developed in [119] in terms of gradations on Clifford modules to a model of differential KO-theory interpreted as "fermionic mass terms". Also develops this model for twisted differential KO-theory.

★ [129] 2023. Land–Nikolaus–Schlichting, *L-theory of C^*-algebras.*
Studies connections between L-theory of real C^*-algebras and topological K-theory (see [128] for previous results along similar lines). Results include:

- There is a unique symmetric monoidal natural transformation (of functors from real C^*-algebras to spectra)
$$\mathrm{ko} \to \mathrm{L}.$$
- For every real C^*-algebra A, the map
$$\mathrm{ko}(A) \otimes_{\mathrm{ko}} \mathrm{L}(\mathbb{R}) \to \mathrm{L}(A)$$
is an equivalence.

★ [50] 2024. Berwick-Evans–Guo, *Power operations preserve Thom classes in twisted equivariant Real K-theory.*
Constructs power operations for twisted KR-theory of topological stacks and shows that these power operations commute with the twisted Atiyah–Bott–Shapiro orientation using algebraic properties of Clifford algebras. Derives corresponding results for KO and KU.

★ [76] 2024. Davis–Wilson, *The cohomology of the connective spectra for K-theory revisited.*
Describes explicit bases for the mod 2 cohomology of connective real and complex K-theory, $\mathrm{H}^*(\mathrm{ko}; \mathbb{Z}/2)$ and $\mathrm{H}^*(\mathrm{ku}; \mathbb{Z}/2)$.

★ [87] 2024. Freed, *Index theory on Pin manifolds.*
An exposition of an index theorem for pin manifolds valued in KO-theory. Also includes a refinement to differential KO-theory, modulo the existence of a proof for the differential KO-index theorem (i.e. the KO-theory version of the main result of [93]). The theorem uses specific inclusions of Pin groups into larger Spin groups to apply the corresponding index

theory for spin manifolds. Also includes a motivating discussion explaining the relevance of pin manifolds in physics. States that this is based on work with Mike Hopkins.

★ [75] 2025. Davis, *The connective KO theory of the Eilenberg-MacLane space $K(\mathbb{Z}/2,2)$.* Computes $\mathrm{ko}_*(K(\mathbb{Z}/2,2))$ and $\mathrm{ko}^*(K(\mathbb{Z}/2,2))$, the connective KO-homology and KO-cohomology of the mod 2 Eilenberg–MacLane space, $K(\mathbb{Z}/2,2)$, using the Adams spectral sequence. Applies the computation to deduce a new result about Stiefel–Whitney classes of Spin manifolds.

6 KR-theory

The references singled out in this section either use or prove things about the particularities of KR-theory in some way (as opposed to KU and KO).

★ [23] 1966. Atiyah, *K-theory and reality.*
Introduces KR as a C_2-equivariant cohomology theory, sometimes called "Real K-theory", built from complex vector bundles with Real structure, that restricts to KO-theory on the subcategory of C_2-spaces with trivial C_2-action. Shows that as an $\mathrm{RO}(C_2)$-graded cohomology theory, KR is $(1,1)$-periodic (i.e. $\mathbb{C}$-periodic). More generally, shows that KR has Thom isomorphisms for Real vector bundles, i.e. complex vector bundles with Real structure. This means that KR has a Real orientation, i.e. a ring map, $\mathrm{MU}_{\mathbb{R}} \to \mathrm{KR}$, from the Real bordism spectrum introduced in [153] .

★ [130] 1967. Landweber, *Fixed point free conjugations on complex manifolds.*

★ [25] 1968. Atiyah, *Bott periodicity and the index of elliptic operators.*
Uses index theory to prove generalizations of the 8-fold Bott periodicity of KO, the 2-fold Bott periodicity of KU, and the $(1,1)$-fold Bott periodicity of KR in the equivariant settings. The most general statement from which all others follow is the following.

- Let $G \to \mathrm{Spin}^c(p,q)$ be a C_2-equivariant homomorphism of groups with C_2-action, and suppose $p \equiv q \mod 8$. Then there is a class $u \in \mathrm{KR}_G(\mathbb{R}^{p,q})$, such that multiplication by u induces an isomorphism,

  $$\mathrm{KR}_G(X) \cong \mathrm{KR}_G(\mathbb{R}^{p,q} \times X).$$

  Here, C_2 acts on $\mathbb{R}^{p,q} = \mathbb{R}^{p+q}$ trivially on the $\mathbb{R}^p$ factor and by -1 on the $\mathbb{R}^q$ factor, and $\mathrm{Spin}^c(p,q)$ is the spin group in the Real Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{R}^{p,q})$ with the induced C_2-action.

The corresponding Thom isomorphisms are formal consequences of the equivariant periodicity theorems.

★ [118] 1968. Karoubi, *Algèbres de Clifford et K-théorie.*
Explores further the relationship between Clifford algebras and K-theory introduced in [26] . Includes both the real and complex cases. need to add more details here.

★ [153] 1968. Landweber, *Conjugations on complex manifolds and equivariant homotopy of MU.*
Introduces the Real bordism spectrum $\mathrm{MU}_{\mathbb{R}} = \mathrm{MR}$, which is a C_2-spectrum whose underlying spectrum is complex bordism, MU, and whose C_2-equivariance arises from complex

conjugation on $\mathrm{MU}(n) =: \mathrm{MU}_{\mathbb{R}}(n,n)$. By the construction of KR-Thom classes for Real vector bundles in [23] , it follows that there is a ring map $\mathrm{MU}_{\mathbb{R}} \to \mathrm{KR}$. Includes partial results on the homotopy groups $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}}$, including:

- The groups $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}}$ are finitely generated and all torsion is of order 2. The torsion subgroup is the kernel of the homomorphism $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}} \to \pi_{p+q}\,\mathrm{MU}$.
- If $p \geq q$, each element of $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}}$ is represented by a map transverse to $\mathrm{BU}(n) \subset \mathrm{MU}(n)$.
- The homomorphism $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}} \to \pi_q\,\mathrm{MO}$ is surjective if $p \leq q$ and zero if $p > q$ (here the second index, q, denotes the trivial q-dimensional representation).
- If $p \not\equiv q \mod 4$, then $\mathrm{MU}_{\mathbb{R}}$ is finite.
- If $p - q \equiv 4 \mod 8$, then the image of $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}} \to \pi_{p+q}\,\mathrm{MU}$ is $2\pi_{p+q}\,\mathrm{MU}$.
- If $p \equiv q \mod 8$, then the image of $\pi_{p,q}\,\mathrm{MU}_{\mathbb{R}} \to \pi_{p+q}\,\mathrm{MU}$ contains $2\pi_{p+q}\,\mathrm{MU} + \mathbb{CP}^1\,\pi_{p+q-2}\,\mathrm{MU}$.
- If $p \equiv q \mod 4$, then $\pi_{p+q}\,\mathrm{MU}_{\mathbb{R}}$ has the same rank as $\pi_{p+q}\,\mathrm{MU}$

★ [33] 1969. Atiyah–Segal, *Equivariant K-theory and completion.*
Proves the "Atiyah–Segal completion theorem" for all compact Lie groups. The theorem concerns the relationship between the equivariant K-theory, $\mathrm{K}^*_G(X)$, of a G-space, X, and the non-equivariant K-theory of its homotopy quotient (a.k.a. Borel construction), $\mathrm{K}^*(X_G) = \mathrm{K}^*(X \times_G \mathrm{E}G)$. In other words, this is the question of whether or not the G-spectrum K_G is "Borel complete", and identifying the corresponding completion map. Results include:

- **Atiyah–Segal completion theorem:** Let G be a compact Lie group and X be a compact G-space. Assume that $\mathrm{KU}^*_G(X)$ is finite over the representation ring $\mathrm{R}(G) = \mathrm{KU}^0_G(*)$. Let I_G be the augmentation ideal in $\mathrm{R}(G)$, and let

$$\mathrm{KU}^*_G(X)^\wedge_{I_G} := \lim_n \Big(\mathrm{KU}^*_G(X)/(I^n_G \cdot \mathrm{KU}^*_G(X)) \Big).$$

  Then the associated bundle construction induces an isomorphism,

$$\mathrm{KU}^*_G(X)^\wedge_{I_G} \cong \mathrm{KU}^*(X_G).$$

- Specializing to the case $X = *$ yields

$$\mathrm{R}(G)^\wedge_{I_G} \cong \mathrm{KU}^0(\mathrm{B}G).$$

- Proves an analogous result for KR-theory, which has implications for KO-theory as well, though they note that the proof does not work if one works entirely with KO-theory. Recovers the computations of [7] .

I don't know whether the finiteness assumption is necessary or not. In [33] , they indicate situations in which it is satisfied, but I often see the condition omitted in other references. In the case $X = *$ and G finite, the completion theorem was proven in [22] , and for $X = *$ and G connected, it was proven in [28] . Note that these statements and proofs did not involve equivariant K-theory.

★ [39] 1971. Atiyah–Singer, *The index of elliptic operators: V.*
Proves a families index theorem for real elliptic operators. This makes essential use of KR-theory.

★ [84] 1971. Edelson, *Real vector bundles and spaces with free involutions.*
Shows that Real vector bundles (in the sense of [23]) are classified by C_2-equivariant homotopy classes of maps into the infinite complex Grassmannian with its complex conjugation C_2-action. Also studies KR-theory of a free C_2-space, and identifies a classifying space for Real line bundles.

★ [96] , 1977. Fujii, *On the relation of real cobordism to KR-theory.*
Uses the Real orientation $\mathrm{MU}_{\mathbb{R}} \to \mathrm{KR}$ of [23] to prove a version of the Conner–Floyd isomorphisms [70] for KR-theory. Results include:

- Let (X, A) be a pair of finite C_2-CW complexes. Then $\mathrm{KR}^{0,0}(X, A)$ is a direct summand of $\mathrm{MU}_{\mathbb{R}}^{0,0}(X, A)$.
- There is an isomorphism
$$\mathrm{MU}_{\mathbb{R}}^{\star}(X, A) \otimes_{\mathrm{MU}_{\mathbb{R}}^{\star}} \mathrm{KR}^{\star} \cong \mathrm{KR}^{\star}(X, A),$$
where $\star$ denotes the $\mathrm{RO}(C_2)$-grading.

★ [122] 1981. Kasparov, *The operator K-functor and extensions of C^*-algebras.*
Introduces bivariant KK-theory for C^*-algebras, along with its product, which gives KK the structure of an additive category. Works in a general setting which applies to the real, complex, and Real cases, as well as the equivariant setting, for G-C^*-algebras. Proves a version of Bott periodicity for KK which is as general as the one in [25] when applied to commutative C^*-algebras.

★ [150] 1982. Nagata–Nishida–Toda, *Segal–Becker theorem for KR-theory.*
Proves an analogue of the results of [164] and [47] for KR-theory. In fact, the results of this paper recover the previous results for KU and KO by forgetting the C_2-action and by taking C_2-fixed points. Results include:

- Let $\mathbb{CP}^\infty$ be the C_2-space given by complex conjugation and let X be a C_2-CW complex. Then there is a split surjection,
$$(\Sigma_+^\infty \mathbb{CP}^\infty)^0(X) \to \mathrm{KR}^0(X),$$

where $\Sigma^\infty_+ \mathbb{CP}^\infty$ is the suspension spectrum in C_2-spectra, and where the map is induced by viewing a rank 1 Real line bundle as a Real virtual bundle.

★ [123] 1988. Kasparov, *Equivariant KK-theory and the Novikov conjecture.*
A follow-up to Kasparov's work on KK-theory [122] . Applies KK-theory to prove the Novikov conjecture in a certain class of cases. more details. The results first appeared in a 1981 preprint which has since been published as [124] in a modified form.

★ [132] 1989. Lawson–Michelsohn, *Spin geometry.*
A book that introduces the basics of spin geometry, including relations with topological K-theory and applications to topology and geometry. Topics include: a detailed introduction to Clifford algebras and their modules, including the Atiyah–Bott–Shapiro construction, which is extended to the KR case using "Real" Clifford modules; a brief introduction to KU, KO, and KR; spinor bundles and Dirac operators; Thom isomorphisms and the Chern character; various versions of the Atiyah–Singer index theorem; and many applications to topology and differential geometry, including metrics of positive scalar curvature and group actions on manifolds. Also contains useful appendices on principal bundles, classifying spaces, characteristic classes, and Thom classes.

★ [85] 1995. Fajstrup, *Tate cohomology of periodic K-theory with reality is trivial.*
Shows that the Tate fixed points of KR vanish. This implies that the homotopy and genuine C_2-fixed points of KR are equivalent,

$$\mathrm{KR}^{hC_2} \simeq \mathrm{KR}^{C_2} \simeq \mathrm{KO}\,.$$

★ [113] 2001. Hu–Kriz, *Real-oriented homotopy theory and an analogue of the Adams–Novikov spectral sequence.*
Shows that the Real orientation of KR, $\mathrm{MU}_{\mathbb{R}} \to \mathrm{KR}$, can be realized as a C_2-$\mathbb{E}_\infty$-map.

★ [81] 2005. Dugger, *An Atiyah–Hirzebruch spectral sequence for KR-theory.*
Develops an analogue of the Atiyah–Hirzebruch spectral sequence [28] in the C_2-equivariant setting of KR-theory, using the equivariant Eilenberg–Mac Lane spectra on the constant Mackey functor, $\underline{\mathbb{Z}}$.

★ [61] 2010. Bruner–Greenlees, *Connective real K-theory of finite groups.*
A book about equivariant connective real K-theory, ko_G and kr_G, which contains both a significant amount of exposition, as well as many computations. Includes:

- the $\mathrm{RO}(C_2)$-graded homotopy groups of kr;
- an equivalence $(\Sigma^{-\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma\,\mathrm{ko}$;

- an equivalence $(\Sigma^{-2\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma^{-6}\,\mathrm{ko}\langle 4\rangle$;
- an equivalence $(\Sigma^{-3\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma^{-5}\,\mathrm{ko}\langle 2\rangle$;
- a cofiber sequence $\mathrm{ko} \to \mathrm{kr}^{hC_2} \to \bigvee_{k\geq 1} \Sigma^{-4k}\,\mathrm{H}\,\mathbb{Z}/2$;
- a cofiber sequence $\mathrm{kr}_{hC_2} \to \mathrm{ko} \to \bigvee_{k\geq 0} \Sigma^{4k}\,\mathrm{H}\,\mathbb{Z}/2$;
- the ko homology, $\mathrm{ko}_*(\mathrm{B}G)$, and cohomology, $\mathrm{ko}^*(\mathrm{B}G)$, of the classifying spaces of various groups, including cyclic groups, generalized quaternion groups, the Klein 4-group, dihedral 2-groups, and elementary abelian 2-groups;

★ [106] 2014. Heard–Stojanoska, *K-theory, reality, and duality.*
Obtains duality results for KO-theory by interpreting it as both the homotopy fixed points and the homotopy orbits of KR. Also includes a discussion of the differentials in the homotopy fixed point spectral sequence for $\mathrm{KR}^{C_2} \simeq \mathrm{KO}$. Results include:

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- The Anderson dual of KR is equivalent to $\Sigma^4\,\mathrm{KR}$, which implies that the Anderson dual of KO is equivalent to $\Sigma^4\,\mathrm{KO} \simeq \mathrm{KSp}$.
- The $K(1)$-local Brown-Comenetz dual of KO is equivalent to $\Sigma^5\,\mathrm{KO}$.

★ [160] 2015. Ricka, *Subalgebras of the $\mathbb{Z}/2$-equivariant Steenrod algebra.*
Computes the C_2-equivariant cohomology of kr, i.e. the connective cover of KR, with coefficients in the constant Mackey functor $\underline{\mathbb{Z}/2}$ as

$$\mathrm{H}^\star(\mathrm{kr};\underline{\mathbb{Z}/2}) \cong \mathcal{A}^\star/\mathcal{E}^\star(2),$$

where $\mathcal{A}^\star$ is the C_2-equivariant mod 2 Steenrod algebra, and $\mathcal{E}^\star(2)$ is a specified subalgebra.

★ [136] 2016. Mathew–Stojanoska, *The Picard group of topological modular forms via descent theory.*
Computes the Picard group $\mathrm{Pic}(\mathrm{KO}) \cong \mathbb{Z}/8$ using C_2-Galois descent for the C_2-Galois extension $\mathrm{KO} \to \mathrm{KU}$, together with the fact that $\mathrm{Pic}(\mathrm{KU}) \cong \mathrm{Pic}(\mathrm{KU}_*) \cong \mathbb{Z}/2$ (see [42]). The computation uses the homotopy fixed point spectral sequence for $\mathfrak{pic}(\mathrm{KU})$ with C_2-action induced from complex conjugation on KU, where $\mathfrak{pic}$ denotes the Picard spectrum.

★ [161] 2016. Rosenberg, *Structure and applications of real C^*-algebras.*

An expository survey of real C^*-algebras, including their K-theory. Emphasizes aspects of the theory and applications that are different for the real case. Topics include metrics of positive scalar curvature, orientifold string theory, and the Baum–Connes conjecture.

★ [142] 2017. Merling, *Equivariant algebraic K-theory of G-rings.*
Introduces a notion of algebraic K-theory for topological G-rings, $\mathrm{K}_G^{\mathrm{top}}$, and shows that

- $\mathrm{K}_G^{\mathrm{top}}(\mathbb{R}, \text{trivial action}) \simeq \mathrm{ko}_G$,
- $\mathrm{K}_G^{\mathrm{top}}(\mathbb{C}, \text{trivial action}) \simeq \mathrm{ku}_G$,
- $\mathrm{K}_{C_2}^{\mathrm{top}}(\mathbb{C}, \text{complex conjugation}) \simeq \mathrm{kr}$,

where the G-rings $\mathbb{R}$ and $\mathbb{C}$ are considered with their topology. Does the same method work to show $\mathrm{K}^{\mathrm{alg}}_{G\rtimes C_2}(\mathbb{C}, \text{trivial} \rtimes \text{complex conjugation}) \simeq \mathrm{kr}_G$, for any Real group G?

★ [147] 2019. Mohsen, *Chern–Simons invariants in KK-theory.*
Realizes the Chern–Simons invariants ([68] , [29] , [30] , [31]) as elements of real KK-theory.

★ [103] 2020. Guillou–Hill–Isaksen–Ravenel, *The cohomology of C_2-equivariant $\mathcal{A}(1)$ and the homotopy of ko_{C_2}.*
Computes the C_2-equivariant mod 2 cohomology of ko_{C_2}, the C_2-equivariant connective cover of KO_{C_2}, and uses it to compute the $\mathrm{RO}(C_2)$-graded homotopy ring of ko_{C_2}. Results include:

- The C_2-fixed point spectrum of ko_{C_2} is $(\mathrm{ko}_{C_2})^{C_2} \simeq \mathrm{ko} \vee \mathrm{ko}$.
- Let σ be the 1-dimensional sign C_2-representation. The C_2-fixed point spectrum of $\Sigma^\sigma \mathrm{ko}_{C_2}$ is $(\Sigma^\sigma \mathrm{ko}_{C_2}) \simeq \mathrm{ko}$.
- The Hopf map induces a C_2-equivariant map $\eta : \Sigma^\sigma \mathrm{ko}_{C_2} \to \mathrm{ko}_{C_2}$, which on C_2-fixed points, induces the map $\eta^{C_2} = (-1, 1) : \mathrm{ko} \to \mathrm{ko} \vee \mathrm{ko}$.
- Complexification $\mathrm{KO}_{C_2} \to \mathrm{KR}$ induces a map on connective covers $c : \mathrm{ko}_{C_2} \to \mathrm{kr}$, which participates in a cofiber sequence,

$$\Sigma^\sigma \mathrm{ko}_{C_2} \xrightarrow{\eta} \mathrm{ko}_{C_2} \xrightarrow{c} \mathrm{kr}\,.$$

  In terms of spectral Mackey functors [cite, also discuss the spectral Mackey functor form of KR somewhere], this cofiber sequence can be written as

$$\begin{tikzcd}
\mathrm{ko} \arrow[r, "(-1,1)"] \arrow[d, bend right, "0"'] & \mathrm{ko}\vee\mathrm{ko} \arrow[r, "\nabla"] \arrow[d, bend right, "\nabla"'] & \mathrm{ko} \arrow[d, bend right, "c"'] \\
\Sigma^1 \mathrm{ko} \arrow[r, "\eta"'] \arrow[u, bend right, "\eta"'] \arrow[loop below, "\text{sign}"] & \mathrm{ko} \arrow[r, "c"'] \arrow[u, bend right, "\Delta"'] \arrow[loop below, "\text{trivial}"] & \mathrm{ku}, \arrow[u, bend right, "r"'] \arrow[loop below, "\text{conj}"]
\end{tikzcd}$$

where $r : \mathrm{ku} \to \mathrm{ko}$ is induced by taking the real vector bundle underlying a complex vector bundle.

- The C_2-equivariant cohomology of ko_{C_2} with coefficients in the constant Mackey functor $\underline{\mathbb{Z}/2}$ is
$$\mathrm{H}^\star(\mathrm{ko}_{C_2}; \underline{\mathbb{Z}/2}) \cong \mathcal{A}^{C_2} /\!/ \mathcal{A}^{C_2}(1),$$
where $\mathcal{A}^{C_2}$ is the C_2-equivariant mod 2 Steenrod algebra.

- The geometric fixed points of ko_{C_2} is $(\mathrm{ko}_{C_2})^{gC_2} \simeq \bigvee_{k\geq 0} \Sigma^{4k}\mathrm{H}\hat{\mathbb{Z}}_2$, where $\hat{\mathbb{Z}}_2$ is the 2-adic integers.

★ [50] 2024. Berwick-Evans–Guo, *Power operations preserve Thom classes in twisted equivariant Real K-theory.*
Constructs power operations for twisted KR-theory of topological stacks and shows that these power operations commute with the twisted Atiyah–Bott–Shapiro orientation using algebraic properties of Clifford algebras. Derives corresponding results for KO and KU.

7 Equivariant K-theory

This section contains references related to the G-spectra KU_G, KO_G, or the $C_2 \ltimes G$-spectrum KR_G for some (Real) compact Lie group, G. In particular, even though KR is itself a C_2-equivariant spectrum, Section 6 is devoted to KR-theory, while the current section is concerned with the presence of an additional group of equivariance, G.

★ [24] 1966. Atiyah, *Power operations in K-theory.*
Systematically studies power operations in KU-theory via taking (nth) tensor powers of vector bundles viewed as elements in (Σ_n-) equivariant KU-theory, which relates K-theory operations to the representation rings of symmetric groups. Compares the Adams operations of [1] to the Steenrod operations via the Atiyah–Hirzebruch spectral sequence and the Chern character [28] .

★ [25] 1968. Atiyah, *Bott periodicity and the index of elliptic operators.*
Uses index theory to prove generalizations of the 8-fold Bott periodicity of KO, the 2-fold Bott periodicity of KU, and the $(1,1)$-fold Bott periodicity of KR in the equivariant settings. The most general statement from which all others follow is the following.

- Let $G \to \mathrm{Spin}^c(p,q)$ be a C_2-equivariant homomorphism of groups with C_2-action, and suppose $p \equiv q \mod 8$. Then there is a class $u \in \mathrm{KR}_G(\mathbb{R}^{p,q})$, such that multiplication by u induces an isomorphism,

  $$\mathrm{KR}_G(X) \cong \mathrm{KR}_G(\mathbb{R}^{p,q} \times X).$$

  Here, C_2 acts on $\mathbb{R}^{p,q} = \mathbb{R}^{p+q}$ trivially on the $\mathbb{R}^p$ factor and by -1 on the $\mathbb{R}^q$ factor, and $\mathrm{Spin}^c(p,q)$ is the spin group in the Real Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{R}^{p,q})$ with the induced C_2-action.

The corresponding Thom isomorphisms are formal consequences of the equivariant periodicity theorems.

★ [35] 1968. Atiyah–Singer, *The index of elliptic operators: I.*
Part I of a five part series on the "Atiyah–Singer index theorem". Reviews Thom classes in KU-theory and KU_G-theory via the exterior algebra construction (see [40] and [165]), along with the induced topological index

$$\text{t-ind} : \mathrm{KU}^0_G(TX) \to \mathrm{KU}^0_G(*) \cong \mathrm{RU}(G),$$

where X is a compact smooth G-manifold. Defines the notion of an *index function* to be a natural transformation $\mathrm{KU}^0_{(-)}(T(-)) \to \mathrm{KU}^0_{(-)}(*) \cong \mathrm{RU}(-)$ of functors between appropriately defined categories that satisfies certain axioms. Shows that there is exactly one index function, equal to t-ind. Reviews facts about psuedo-differential operators and elliptic operators over X, and defines the symbol of an elliptic operator as an element of $\mathrm{KU}^0_G(TX)$. Defines the index of a G-invariant[7] elliptic operator P over X as

$$[\mathrm{Ker}(P)] - [\mathrm{Coker}(P)] \in \mathrm{KU}^0_G(*) \cong \mathrm{RU}(G),$$

and explains how the index of P only depends on its symbol in $\mathrm{KU}_G(TX)$. Thus, the index defines a map

$$\text{a-ind} : \mathrm{KU}^0_G(TX) \to \mathrm{KU}^0_G(*) \cong \mathrm{RU}(G),$$

called the analytic index. Shows that a-ind is an index function, thus showing that it equals t-ind,

$$\text{a-ind} = \text{t-ind} : \mathrm{KU}^0_G(TX) \to \mathrm{KU}^0_G(*).$$

As a consequence, a topological formula is obtained for the index of an elliptic operator on a G-manifold in terms of topological K-theory.

★ [165] 1968. Segal, *Equivariant K-theory.*
An introduction to equivariant K-theory built from G-equivariant vector bundles. This work is Segal's thesis, and much of it is attributed to Atiyah.

★ [33] 1969. Atiyah–Segal, *Equivariant K-theory and completion.*
Proves the "Atiyah–Segal completion theorem" for all compact Lie groups. The theorem concerns the relationship between the equivariant K-theory, $\mathrm{K}^*_G(X)$, of a G-space, X, and the non-equivariant K-theory of its homotopy quotient (a.k.a. Borel construction), $\mathrm{K}^*(X_G) = \mathrm{K}^*(X \times_G \mathrm{E}G)$. In other words, this is the question of whether or not the G-spectrum K_G is "Borel complete", and identifying the corresponding completion map. Results include:

- **Atiyah–Segal completion theorem:** Let G be a compact Lie group and X be a compact G-space. Assume that $\mathrm{KU}^*_G(X)$ is finite over the representation ring $\mathrm{R}(G) = \mathrm{KU}^0_G(*)$. Let I_G be the augmentation ideal in $\mathrm{R}(G)$, and let

  $$\mathrm{KU}^*_G(X)^\wedge_{I_G} := \lim_n \Big(\mathrm{KU}^*_G(X)/(I^n_G \cdot \mathrm{KU}^*_G(X)) \Big).$$

  Then the associated bundle construction induces an isomorphism,

  $$\mathrm{KU}^*_G(X)^\wedge_{I_G} \cong \mathrm{KU}^*(X_G).$$

- Specializing to the case $X = *$ yields

  $$\mathrm{R}(G)^\wedge_{I_G} \cong \mathrm{KU}^0(\mathrm{B}G).$$

- Proves an analogous result for KR-theory, which has implications for KO-theory as well, though they note that the proof does not work if one works entirely with KO-theory. Recovers the computations of [7] .

[7] elliptic operators which are not G-invariant can be replaced with a G-invariant one with the same symbol.

I don't know whether the finiteness assumption is necessary or not. In [33] , they indicate situations in which it is satisfied, but I often see the condition omitted in other references. In the case $X = *$ and G finite, the completion theorem was proven in [22] , and for $X = *$ and G connected, it was proven in [28] . Note that these statements and proofs did not involve equivariant K-theory.

★ [137] 1971. Matumoto, *Equivariant K-theory and Fredholm operators.*
Generalizes the Atiyah–Jänich theorem (Appendix of [40] , and [114]) to the equivariant setting. Results include:

- Let G be a compact Lie group, and let $\mathcal{F}$ be the G-space of Fredholm operators on the G-Hilbert space $L^2(G, \ell^2)$. Then for any compact G-space X, the G-index map induces an isomorphism,

$$\text{index} : [X, \mathcal{F}]_G \xrightarrow{\sim} \mathrm{KU}^0_G(X).$$

For the analogous generalization of [37] in higher degrees, see [FHT?].

★ [166] 1971. Segal, *Equivariant stable homotopy theory.*
Introduces $\mathrm{RO}(G)$-graded equivariant stable homotopy theory for a finite group G, as well as an equivariant J-homomorphism, $J : \mathrm{KO}^{-1}_G(X) \to \mathrm{gl}_1\, \mathbb{S}^0_G(X)$. Identifies the equivariant stable cohomotopy ring of a point (in degree 0) as the Burside ring, $\mathrm{A}(G)$, of the group G:

$$\mathbb{S}^0_G(*) = \pi_0\, \mathbb{S}_G := \operatorname*{colim}_V [S^V, S^V]_G \cong \mathrm{A}(G),$$

where V is a real G-representation, and $\mathrm{A}(G)$ is the Grothendieck group of the category of finite G-sets.

★ [138] 1973. Matumoto, *Equivariant cohomology theories on G-CW copmlexes.*
Develops an Atiyah–Hirzebruch spectral sequence [28] for equivariant cohomology theories, and specifies conditions under which the spectral sequence for KU_G collapses. Applies the spectral sequence to compute KU^*_G of certain types of $\mathrm{O}(n)$-manifolds.

★ [122] 1981. Kasparov, *The operator K-functor and extensions of C^*-algebras.*
Introduces bivariant KK-theory for C^*-algebras, along with its product, which gives KK the structure of an additive category. Works in a general setting which applies to the real, complex, and Real cases, as well as the equivariant setting, for G-C^*-algebras. Proves a version of Bott periodicity for KK which is as general as the one in [25] when applied to commutative C^*-algebras.

★ [123] 1988. Kasparov, *Equivariant KK-theory and the Novikov conjecture.*
A follow-up to Kasparov's work on KK-theory [122] . Applies KK-theory to prove the Novikov conjecture in a certain class of cases. more details. The results first appeared in a 1981 preprint which has since been published as [124] in a modified form.

★ [97] 1994. Getzler, *The equivariant Chern character for non-compact Lie groups.*
Constructs a model for the equivariant Chern character of an equivariant vector bundle along the lines of the Chern–Weil construction in the non-equivariant case.

★ [124] 1995. Kasparov, *K-theory, group C^*-algebras, and higher signatures (Conspectus) (first distributed 1981).*
A (modified) preprint version of [123] .

★ [102] 2004. Greenlees, *Equivariant connective K-theory for compact Lie groups.*
Constructs a G-commutative ring spectrum ku_G, for any compact Lie group G, such that

1. the underlying spectrum of ku_G is ku;
2. there is a ring map $\mathrm{MU}_G \to \mathrm{ku}_G$; i.e. ku_G is complex orientable;
3. there is a ring map $\mathrm{ku}_G \to \mathrm{KU}_G$ which is the localization that inverts the Bott element;
4. the coefficient ring ku_G^* is Noetherian, and it's Krull dimension is
$$\max\{2, \text{maximal rank of an abelian subgroup of } G\};$$
5. an analogue of the Atiyah–Segal completion theorem [33] holds, namely,
$$\mathrm{ku}^*(\mathrm{B}G) \cong \mathrm{ku}_G^*(\mathrm{E}G) \cong (\mathrm{ku}_G^*)^\wedge_I,$$
where $I = \ker(\mathrm{ku}_G^* \to \mathrm{ku}^*)$ is the augmentation ideal.

Also contains partial computations of the coefficient ring ku_G^* for certain groups.

★ [116] 2004. Joachim, *Higher coherences for equivariant K-theory.*
Constructs the spin^c orientation, $\mathrm{MSpin}^c \to \mathrm{KU}$, of [26] as a map of $\mathbb{E}_\infty$-ring spectra, using a C^*-algebra model of K-theory and an orthogonal spectra model of $\mathbb{E}_\infty$-ring spectra. The results are actually obtained in the generality of equivariant K-theory, KU_G, and the methods apply to obtain the spin orientation, $\mathrm{MSpin} \to \mathrm{KO}$ [26] , as an $\mathbb{E}_\infty$-map as well. The equivariant spin^c orientations constructed here are shown to refine to a map of global orthogonal spectra in [53] .

★ [152] 2009. Ortiz, *Differential Equivariant K-Theory.*
Constructs various models of differential equivariant KU-theory for a finite group.

★ [61] 2010. Bruner–Greenlees, *Connective real K-theory of finite groups.*
A book about equivariant connective real K-theory, ko_G and kr_G, which contains both a significant amount of exposition, as well as many computations. Includes:

- the $\mathrm{RO}(C_2)$-graded homotopy groups of kr;

- an equivalence $(\Sigma^{-\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma\,\mathrm{ko}$;
- an equivalence $(\Sigma^{-2\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma^{-6}\,\mathrm{ko}\langle 4\rangle$;
- an equivalence $(\Sigma^{-3\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma^{-5}\,\mathrm{ko}\langle 2\rangle$;
- a cofiber sequence $\mathrm{ko} \to \mathrm{kr}^{hC_2} \to \bigvee_{k\geq 1} \Sigma^{-4k}\,\mathrm{H}\,\mathbb{Z}/2$;
- a cofiber sequence $\mathrm{kr}_{hC_2} \to \mathrm{ko} \to \bigvee_{k\geq 0} \Sigma^{4k}\,\mathrm{H}\,\mathbb{Z}/2$;
- the ko homology, $\mathrm{ko}_*(\mathrm{B}G)$, and cohomology, $\mathrm{ko}^*(\mathrm{B}G)$, of the classifying spaces of various groups, including cyclic groups, generalized quaternion groups, the Klein 4-group, dihedral 2-groups, and elementary abelian 2-groups;

★ [53] 2014. Bohmann, *Global orthogonal spectra.*
Gives a definition of global orthogonal spectra, and shows that the equivariant Atiyah–Bott–Shapiro [26] spin^c orientations of KU_G constructed in [116] refine to a map of global orthogonal spectra, $\mathrm{MSpin}^c \to \mathrm{KU}$.

★ [160] 2015. Ricka, *Subalgebras of the $\mathbb{Z}/2$-equivariant Steenrod algebra.*
Computes the C_2-equivariant cohomology of kr, i.e. the connective cover of KR, with coefficients in the constant Mackey functor $\underline{\mathbb{Z}/2}$ as

$$\mathrm{H}^\star(\mathrm{kr};\underline{\mathbb{Z}/2}) \cong \mathcal{A}^\star/\mathcal{E}^\star(2),$$

where $\mathcal{A}^\star$ is the C_2-equivariant mod 2 Steenrod algebra, and $\mathcal{E}^\star(2)$ is a specified subalgebra.

★ [142] 2017. Merling, *Equivariant algebraic K-theory of G-rings.*
Introduces a notion of algebraic K-theory for topological G-rings, $\mathrm{K}_G^{\mathrm{top}}$, and shows that

- $\mathrm{K}_G^{\mathrm{top}}(\mathbb{R}, \text{trivial action}) \simeq \mathrm{ko}_G$,
- $\mathrm{K}_G^{\mathrm{top}}(\mathbb{C}, \text{trivial action}) \simeq \mathrm{ku}_G$,
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★ [147] 2019. Mohsen, *Chern–Simons invariants in KK-theory.*
Realizes the Chern–Simons invariants ([68] , [29] , [30] , [31]) as elements of real KK-theory.

★ [103] 2020. Guillou–Hill–Isaksen–Ravenel, *The cohomology of C_2-equivariant $\mathcal{A}(1)$ and the homotopy of ko_{C_2}.*
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- Complexification $\mathrm{KO}_{C_2} \to \mathrm{KR}$ induces a map on connective covers $c : \mathrm{ko}_{C_2} \to \mathrm{kr}$, which participates in a cofiber sequence,

  $$\Sigma^\sigma \mathrm{ko}_{C_2} \xrightarrow{\eta} \mathrm{ko}_{C_2} \xrightarrow{c} \mathrm{kr}\,.$$

  In terms of spectral Mackey functors [cite, also discuss the spectral Mackey functor form of KR somewhere], this cofiber sequence can be written as

  $$\begin{tikzcd}
\mathrm{ko} \arrow[r, "{(-1,1)}"] \arrow[d, bend right, "0"'] & \mathrm{ko} \vee \mathrm{ko} \arrow[r, "\nabla"] \arrow[d, bend right, "\nabla"'] & \mathrm{ko} \arrow[d, bend right, "c"'] \\
\Sigma^1 \mathrm{ko} \arrow[r, "\eta"'] \arrow[u, bend right, "\eta"'] \arrow[loop below, "\mathrm{sign}"] & \mathrm{ko} \arrow[r, "c"'] \arrow[u, bend right, "\Delta"'] \arrow[loop below, "\mathrm{trivial}"] & \mathrm{ku}, \arrow[u, bend right, "r"'] \arrow[loop below, "\mathrm{conj}"]
\end{tikzcd}$$

  where $r : \mathrm{ku} \to \mathrm{ko}$ is induced by taking the real vector bundle underlying a complex vector bundle.
- The C_2-equivariant cohomology of ko_{C_2} with coefficients in the constant Mackey functor $\underline{\mathbb{Z}/2}$ is

  $$\mathrm{H}^\star(\mathrm{ko}_{C_2}; \underline{\mathbb{Z}/2}) \cong \mathcal{A}^{C_2} /\!/ \mathcal{A}^{C_2}(1),$$

  where $\mathcal{A}^{C_2}$ is the C_2-equivariant mod 2 Steenrod algebra.
- The geometric fixed points of ko_{C_2} is $(\mathrm{ko}_{C_2})^{gC_2} \simeq \bigvee_{k \geq 0} \Sigma^{4k} \mathrm{H}\hat{\mathbb{Z}}_2$, where $\hat{\mathbb{Z}}_2$ is the 2-adic integers.

★ [45] 2022. Balderrama, *K-theory equivariant with respect to an elementary abelian 2-group.*
Computes the $\mathrm{RO}(G)$-graded homotopy Mackey functor of KU_G and KO_G for $G = (\mathbb{Z}/2)^n$, together with product structures and power operations.

★ [50] 2024. Berwick-Evans–Guo, *Power operations preserve Thom classes in twisted equivariant Real K-theory.*
Constructs power operations for twisted KR-theory of topological stacks and shows that these power operations commute with the twisted Atiyah–Bott–Shapiro orientation using algebraic properties of Clifford algebras. Derives corresponding results for KO and KU.

8 Twisted K-theory

★ [176] 1964. Wall, *Graded Brauer groups.*
Shows that the set of equivalence classes of $\mathbb{Z}/2$-graded central simple algebras over a field k forms a group, called the *($\mathbb{Z}/2$-) graded Brauer group* (or *Brauer–Wall group*) of k, and shows that the $\mathbb{Z}/2$-graded Brauer group of $\mathbb{R}$ is $\mathbb{Z}/8$ and that of $\mathbb{C}$ is $\mathbb{Z}/2$, in both cases represented by the corresponding type of Clifford algebras under tensor product. The results are obtained for fields k more general than only $\mathbb{R}$ and $\mathbb{C}$.

★ [79] 1970. Donovan–Karoubi, *Graded Brauer groups and K-theory with local coefficients.*
Defines twisted K-theory groups, $\mathrm{KU}^{\alpha}(X)$ and $\mathrm{KO}^{\beta}(X)$, for twists,

$$\alpha \in \mathbb{Z}/2 \times \mathrm{H}^1(X;\mathbb{Z}/2) \times \mathrm{H}^3(X;\mathbb{Z})_{\mathrm{tors}},$$

and

$$\beta \in \mathbb{Z}/8 \times \mathrm{H}^1(X;\mathbb{Z}/2) \times \mathrm{H}^2(X;\mathbb{Z}/2),$$

where A_{tors} denotes the torsion subgroup of an abelian group A. (What is the space $K(A,n)_{\mathrm{tors}}$ that classifies $\mathrm{H}^n(-;A)_{\mathrm{tors}}$? Or does it even exist?) The groups classifying the twists α and β over a space X arise from considering the $\mathbb{Z}/2$-graded Brauer groups of bundles of (real or complex) $\mathbb{Z}/2$-graded central simple algebras on X. When X is a point, these groups are given by the Morita classes of Clifford algebras ([26] , [176]). Compare the groups of twists above to the results of [133] ; namely, the twists for K-theory considered here are some, but not all, of the possible twists of K-theory. Generalizes the Thom isomorphism in K-theory ([40] , [26]) to all real vector bundles, where the degree shift must now include non-integral twists. This was the initial motivation for considering the types of twists that they did. Also includes a discussion of the multiplicative structure and Adams operations [1] in twisted K-theory. See [50] for results on the relationship between power operations and Thom classes in the general context of twisted equivariant KR-theory of topological stacks.

★ [133] 1977. Madsen–Snaith–Tornehave, *Infinite loop maps in geometric topology.*
Shows that there are equivalences of infinite loop spaces,

$$\mathrm{GL}_1\,\mathrm{KU} \simeq K(\mathbb{Z}/2,0) \times K(\mathbb{Z},2) \times \mathrm{BSU}_{\otimes},$$

and

$$\mathrm{GL}_1\,\mathrm{KO} \simeq K(\mathbb{Z}/2,0) \times K(\mathbb{Z}/2,1) \times \mathrm{BSO}_{\otimes}.$$

These results classify (one connected component of) twists of K-theory by the spaces

$$\mathrm{BGL}_1\,\mathrm{KU} \simeq K(\mathbb{Z}/2,1) \times K(\mathbb{Z},3) \times \mathrm{BBSU}_{\otimes},$$

and

$$\mathrm{BGL}_1\,\mathrm{KO} \simeq K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2) \times \mathrm{BBSO}_{\otimes}.$$

★ [21] 2004. Atiyah–Segal, *Twisted K-theory.*
Constructs a version of twisted KU-theory for twists classified by $K(\mathbb{Z}, 3)$ in terms of families of Fredholm operators on Hilbert bundles

★ [15] 2008. Ando–Blumberg–Gepner–Hopkins–Rezk, *Units of ring spectra and Thom spectra.*
An older preprint version of [17] and [14] . Develops the theory of units of ring spectra and their relationship with Thom spectra using two approaches: ∞-categories and rigid infinite loop space theory. This theory lays down the theoretical homotopical underpinnings of twisted K-theory. For its application to twisted K-theory, see [13] . This theory is also applied in [16] to identify the space of $\mathbb{E}_\infty$-orientations $\mathrm{MSpin} \to \mathrm{KO}$.

★ [13] 2010. Ando–Blumberg–Gepner, *Twists of K-theory and TMF.*
Relates the version of twisted K-theory in terms of Fredholm operators [21] to the homotopy theoretic approach developed in [15] , [17] , and [14] . Extends these ideas to construct twists of elliptic cohomology classified by $K(\mathbb{Z}, 4)$.

★ [17] 2014. Ando–Blumberg–Gepner–Hopkins–Rezk, *An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology.*
One of two published versions of [15] developing the theory of units of ring spectra and their relationship with Thom spectra via ∞-category theory. This theory lays down the theoretical homotopical underpinnings of twisted K-theory. For its application to twisted K-theory, see [13] .

★ [14] 2014. Ando–Blumberg–Gepner–Hopkins–Rezk, *Units of ring spectra, orientations, and Thom spectra via rigid infinite loop space theory.*
One of two published versions of [15] developing the theory of units of ring spectra and their relationship with Thom spectra via rigid infinite loop space theory. This theory lays down the theoretical homotopical underpinnings of twisted K-theory. For its application to twisted K-theory, see [13] .

★ [18] 2014. Antieau–Gepner–Gómez, *Actions of $K(\pi, n)$ spaces on K-theory and uniqueness of twisted K-theory.*
Proves that

- $[K(\mathbb{Z}, 3), \mathrm{BGL}_1 \mathrm{KU}] \cong [K(\mathbb{Z}, 3), K(\mathbb{Z}, 3)] \cong \mathbb{Z}$, and
- $[K(\mathbb{Z}/2, 2), \mathrm{BGL}_1 \mathrm{KO}] \cong [K(\mathbb{Z}/2, 2), K(\mathbb{Z}/2, 2)] \cong \mathbb{Z}/2$.

This allows one to compare various geometric definitions of twisted KU-theory (resp. KO-theory) twisted by elements of $\mathrm{H}^3(-; \mathbb{Z})$ (resp. $\mathrm{H}^2(-; \mathbb{Z}/2)$) via the homotopy class of the corresponding map $K(\mathbb{Z}, 3) \to \mathrm{BGL}_1 \mathrm{KU}$ (resp. $K(\mathbb{Z}/2, 2) \to \mathrm{BGL}_1 \mathrm{KO}$). In particular, in the real case, all nontrivial ways of defining twisted KO-theory for twists classified by

$K(\mathbb{Z}/2, 2)$ are equivalent. Also contains an appendix which explains how to adapt the geometric model for KO-theory given in [115] to the twisted case, within the general framework for twisted cohomology set up in [15] , [14] , and [17] .

★ [74] 2015. Dadarlat–Pennig, *Unit spectra of K-theory from strongly self-absorbing C^*-algebras.*
Interprets the group $\mathrm{gl}_1(\mathrm{KU})^1(X)$ as the group of isomorphism classes of locally trivial bundles of C^*-algebras whose fiber is isomorphic to the stable Cuntz algebra, under tensor product. Also provides a similar description of $\mathrm{gl}_1(\mathrm{KU}_{(p)})^1(X)$.

★ [155] 2016. Pennig, *A non-commutative model for higher twisted K-theory.*
Models twisted KU-theory for arbitrary twists classified by $\mathrm{BGL}_1(\mathrm{KU})$ in terms of strongly self-absorbing C^*-algebras.

★ [100] 2023. Gomi–Yamashita, *Differential KO-theory via gradations and mass terms.*
Refines the model of KO-theory developed in [119] in terms of gradations on Clifford modules to a model of differential KO-theory interpreted as “fermionic mass terms”. Also develops this model for twisted differential KO-theory.

★ [50] 2024. Berwick-Evans–Guo, *Power operations preserve Thom classes in twisted equivariant Real K-theory.*
Constructs power operations for twisted KR-theory of topological stacks and shows that these power operations commute with the twisted Atiyah–Bott–Shapiro orientation using algebraic properties of Clifford algebras. Derives corresponding results for KO and KU.

9 Differential K-theory

★ [158] 1985. Quillen, *Superconnections and the Chern character.*
Introduces the notion of a superconnection on a super ($\mathbb{Z}/2$-graded) vector bundle, and uses it to construct families of differential form representatives for the Chern character. Includes a corresponding geometric description of the Chern character in odd degrees using Clifford linear superconnections. See [135] for a refinement of the Riemann–Roch theorem [27] to an equality of differential forms using this model of the Chern character.

★ [135] 1986. Mathai–Quillen, *Superconnections, Thom classes, and equivariant differential forms.*
Proves a refinement of the differentiable Riemann–Roch theorem [27] to an equality of differential form representatives using the Chern character form constructed in [158] and the Thom classes constructed in [26] .

★ [97] 1994. Getzler, *The equivariant Chern character for non-compact Lie groups.*
Constructs a model for the equivariant Chern character of an equivariant vector bundle along the lines of the Chern–Weil construction in the non-equivariant case.

★ [86] 2000. Freed, *Dirac charge quantization and generalized differential cohomology.*
Discusses physical motivations for generalized differential cohomology. Argues that currents associated to quantized charges for an abelian gauge field should be classified by a generalized cohomology theory equipped with a map to real valued cohomology, $c : E \to \mathrm{H}\mathbb{R}$, together with the additional data of differential forms that are compatible with the map c via de Rham cohomology, in an appropriate sense. See [112] for the underlying mathematical theory of generalized differential cohomology that was being developed at around the same time.

★ [88] 2000. Freed–Hopkins, *On Ramond-Ramond fields and K-theory.*
Builds off of [148] 's observation that Ramond-Ramond fields in type-II string theory have topological invariants in K-theory by suggesting that Ramond-Ramond fields are classified by *differential* K-theory, thus leading to the development of differential K-theory in the first place.

★ [112] 2005. Hopkins–Singer, *Quadratic functions in geometry, topology, and M-theory.*
Constructs differential refinements for arbitrary cohomology theories, and includes a brief discussion about differential K-theory. Includes a discussion of Anderson duality (also see [106]). Add details.

★ [95] 2007. Freed–Moore–Segal, *The uncertainty of fluxes.*
Studies the Anderson duality of KO, as in [112] and [106] .

★ [65] 2009. Bunke–Schick, *Smooth K-theory.*
Constructs a multiplicative differential refinement, $\widehat{\mathrm{KU}}$, of KU-theory using geometric cocycles, together with pushforward maps, $p_! : \widehat{\mathrm{KU}}(W) \to \widehat{\mathrm{KU}}(B)$, associated to KU-orientable proper submersions $p : W \to B$, and refines the Chern character to a natural transformation $\widehat{\mathrm{ch}} : \widehat{\mathrm{KU}} \to \widehat{\mathrm{H}\mathbb{Q}}$. Proves a refinement of the Atiyah–Singer index theorem [?] regarding the compatibility of pushforwards with the Chern character.

★ [152] 2009. Ortiz, *Differential Equivariant K-Theory.*
Constructs various models of differential equivariant KU-theory for a finite group.

★ [62] 2010. Bunke, *Adams operations in smooth K-theory.*
Constructs refinements of the Adams operations [1] for differential (a.k.a. smooth) KU-theory, and proves a corresponding compatibility theorem between the differential Adams operations and pushforward maps for KU-oriented bundles.

★ [66] 2010. Bunke–Schick, *Uniqueness of smooth extensions of generalized cohomology theories.*
Introduces axioms for differential refinements (a.k.a. smooth extensions), $\widehat{E}$, of cohomology theories, E, and proves a corresponding uniqueness theorem for such refinements under certain extra conditions (including the presence of additional structure, such as "integration" or a multiplication). In particular, the differential cohomology theories constructed in [112] satisfy the axioms, and KU and KO satisfy the additional conditions of the uniqueness theorem, implying that any two multiplicative differential refinements of $\widehat{\mathrm{KU}}$ or $\widehat{\mathrm{KO}}$ that satisfy the axioms are isomorphic. They also give examples of non-isomorphic differential refinements of KU-theory in the absence of a multiplicative structure or integration.

★ [93] 2010. Freed–Lott, *An index theorem in differential K-theory.*
Refines the Atiyah–Singer index theorem [?] to differential KU-theory. More specifically, for a proper submersion $\pi : X \to B$ of relative dimension n with a Riemannian structure and a differential spin^c-structure, they construct analytic and topological pushforward maps $\mathrm{ind}^{\mathrm{an}}, \mathrm{ind}^{\mathrm{top}} : \widehat{\mathrm{KU}}^0(X) \to \widehat{\mathrm{KU}}^{-n}(B)$, and show that $\mathrm{ind}^{\mathrm{an}} = \mathrm{ind}^{\mathrm{top}}$.

★ [63] 2012. Bunke, *Differential cohomology.*
Lecture notes for a course on differential cohomology. Starts with ordinary differential cohomology, and proceeds to develop generalized differential cohomology (which includes the constructions of [112]) using the machinery of ∞-categories. Includes a detailed discussion of differential KU-theory.

★ [67] 2012. Bunke–Schick, *Differential K-theory: A survey.*
An expository survey of differential KU-theory. Discusses various models, an axiomatic characterization, and analogs of the Riemann–Roch and Atiyah–Singer index theorems for

differential KU-theory.

★ [100] 2023. Gomi–Yamashita, *Differential KO-theory via gradations and mass terms.* Refines the model of KO-theory developed in [119] in terms of gradations on Clifford modules to a model of differential KO-theory interpreted as "fermionic mass terms". Also develops this model for twisted differential KO-theory.

★ [87] 2024. Freed, *Index theory on Pin manifolds.* An exposition of an index theorem for pin manifolds valued in KO-theory. Also includes a refinement to differential KO-theory, modulo the existence of a proof for the differential KO-index theorem (i.e. the KO-theory version of the main result of [93]). The theorem uses specific inclusions of Pin groups into larger Spin groups to apply the corresponding index theory for spin manifolds. Also includes a motivating discussion explaining the relevance of pin manifolds in physics. States that this is based on work with Mike Hopkins.

★ [101] 2025. Grady, *On the differential K-theory of moduli stacks.* Computes the connective differential K-theory and the differential cohomology of the moduli stack of principal G-bundles with connection.

10 C^*-algebras

★ [20] 1948. Arens–Kaplansky, *Topological representation of algebras.*
Commutative real C^*-algebras are classified by locally compact Hausdorff C_2-spaces via the assignment $(X, \tau) \mapsto C_0(X, \tau) = \{f \in C_0(X) \mid f(\tau x) = \overline{f(x)}\}$.

★ [60] 1977. Brown–Green–Rieffel, *Stable isomorphism and strong Morita equivalence of C^*-algebras.*
Shows that if A and B are C^*-algebras with countable approximate identities, then A and B are strongly Morita equivalent if and only if they are stably isomorphic. maybe expand these notions in detail here.

★ [80] 1977. Douglas, *Extensions of C^*-algebras and K-homology.*
Part of the book [149] . Describes a homology theory, $\mathrm{Ext}_k(X)$, where $\mathrm{Ext}_1(X)$ is a group of equivalence classes of extensions of the C^*-algebra, $\mathcal{K}$, of compact operators on a separable Hilbert space, by the C^*-algebra, $C(X)$, of continuous complex valued functions on X. Explains that Ext_* is the homology theory dual to KU^* by exhibiting a homomorphism $\mathrm{Ext}_1(X) \to \mathrm{Hom}(\mathrm{KU}^1(X), \mathbb{Z})$, using an index construction. Also explains how $\mathrm{Ext}_1(X)$ carries the structure of a $\mathrm{KU}^0(X)$-module.

★ [149] 1977. Morrel–Singer (Eds.), *K-theory and operator algebras.*
A collection of talks from a conference held at the University of Georgia, April 21 – April 25, 1975, on connections between K-theory and operator algebras. Most of the talks are both expository and exploratory in nature. Contains the references: [80] , [140] , [167] .

★ [122] 1981. Kasparov, *The operator K-functor and extensions of C^*-algebras.*
Introduces bivariant KK-theory for C^*-algebras, along with its product, which gives KK the structure of an additive category. Works in a general setting which applies to the real, complex, and Real cases, as well as the equivariant setting, for G-C^*-algebras. Proves a version of Bott periodicity for KK which is as general as the one in [25] when applied to commutative C^*-algebras.

★ [72] 1984. Cuntz, *K-theory and C^*-algebras.*
An exposition of C^*-algebraic K-theory. Also includes a proof of a generalization of complex Bott periodicity that applies to a class of functors on C^*-algebras based on a homotopy equivalence between the Toeplitz algebra and the complex numbers.

★ [51] 1986. Blackadar, *K-theory for operator algebras.*
A book covering K-theory from the perspective of operator algebras.

★ [73] 1987. Cuntz, *A new look at KK-theory.*

Describes the groups $\mathrm{KK}(A, B)$ in terms of homotopy classes of generalized $*$-homomorphisms from A to B. Specifically, let $qA = \ker(A * A \xrightarrow{m} A)$, where $*$ denotes the coproduct (i.e. free product) of C^*-algebras, and m is the fold map, and let $\mathcal{K}$ be the algebra of compact operators. Then for any (separable?) C^*-algebras A, B,

$$\mathrm{KK}(A, B) \cong \pi_0 \operatorname{Hom}(qA, \mathcal{K} \otimes A),$$

where Hom denotes the space of $*$-homomorphisms.

★ [107] 1987. Higson, *A characterization of KK-theory.*
Shows that KK-theory is the initial additive invariant of C^*-algebras that is stable[8], split exact, and homotopy invariant.

★ [163] 1987. Rosenberg–Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor.*
Proves a Künneth theorem and a universal coefficient theorem for Kasparov's KK-theory [122] for C^*-algebras. Let $\mathcal{N}$ be the smallest subcategory of the category of separable nuclear C^*-algebras that satisfies the following properties:

1. $\mathcal{N}$ contains all separable C^*-algebras of Type I,
2. $\mathcal{N}$ is closed under strong Morita equivalence (equivalently, stable isomorphism [60]), inductive limits, extensions, and crossed products by $\mathbb{R}$ and $\mathbb{Z}$,
3. if $J \subset A$ is an ideal, then $A, J \in \mathcal{N} \implies A/J \in \mathcal{N}$, and $A, A/J \in \mathcal{N} \implies J \in \mathcal{N}$.

Let $A \in \mathcal{N}$, and let B be a C^*-algebra with a countable approximate unit. Then

- if $\mathrm{K}_*(B)$ is finitely generated, then there is a short exact sequence,
$$0 \to \mathrm{K}^*(A) \otimes \mathrm{K}_*(B) \to \mathrm{KK}_*(A, B) \to \operatorname{Tor}_1^{\mathbb{Z}}(\mathrm{K}^*(A), \mathrm{K}_*(B)) \to 0,$$
- there is a short exact sequence,
$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \to \mathrm{KK}_*(A, B) \to \operatorname{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \to 0,$$

where both short exact sequences are natural in each variable and both split unnaturally.

★ [108] 1988. Higson, *Algebraic K-theory of stable C*-algebras.*
Proves (a conjecture of Karoubi) that for any unital C^*-algebra, A, the algebraic and topological K-theory groups of $A \otimes (\mathcal{B}/\mathcal{K})$ agree, where $\mathcal{B}/\mathcal{K}$ is the Calkin algebra (i.e. $\mathcal{B}$ is the algebra of bounded operators on a separable Hilbert space, and $\mathcal{K}$ is the algebra of compact operators). See also [173] .

[8]Here, *stable* means invariant under tensoring with the algebra of compact operators on a separable Hilbert space.

★ [123] 1988. Kasparov, *Equivariant KK-theory and the Novikov conjecture.*
A follow-up to Kasparov's work on KK-theory [122] . Applies KK-theory to prove the Novikov conjecture in a certain class of cases. more details. The results first appeared in a 1981 preprint which has since been published as [124] in a modified form.

★ [109] 1990. Higson, *Categories of fractions and excision in KK-theory.*
Introduces E-theory of C^*-algebras as a close approximation of KK-theory that satisfies excision. Provides a universal property for E-theory that is similar to the one for KK-theory given in [107] .

★ [173] 1992. Suslin–Wodzicki, *Excision in algebraic K-theory.*
Shows that all C^*-algebras satisfy (an algebraic notion of) excision in algebraic K-theory and uses this to prove Karoubi's conjecture [cite] that the algebraic and topological K-theory of stable C^*-algebras agree. Results include: Let $\mathrm{K}^{\mathrm{alg}}$ denote algebraic K-theory, and let $\mathrm{K}^{\mathrm{top}}$ denote topological K-theory (i.e. K-theory of C^*-algebras, where the topologies of the algebras and their spaces of automorphisms are taken into account).

- For any C^*-algebra, A,

$$\mathrm{K}^{\mathrm{top}}(A) \cong \mathrm{K}^{\mathrm{top}}(A \otimes \mathcal{K}) \cong \mathrm{K}^{\mathrm{alg}}(A \otimes \mathcal{K}),$$

  where $\mathcal{K}$ is the algebra of compact operators on an infinite dimensional separable Hilbert space.

Apply this to obtain KU, KO, and maybe KR as *algebraic* K-theory of particular rings.

★ [124] 1995. Kasparov, *K-theory, group C^*-algebras, and higher signatures (Conspectus) (first distributed 1981).*
A (modified) preprint version of [123] .

★ [52] 2002. Boersema, *Real C^*-algebras, united K-theory, and the Künneth formula.*

★ [116] 2004. Joachim, *Higher coherences for equivariant K-theory.*
Constructs the spin^c orientation, $\mathrm{MSpin}^c \to \mathrm{KU}$, of [26] as a map of $\mathbb{E}_\infty$-ring spectra, using a C^*-algebra model of K-theory and an orthogonal spectra model of $\mathbb{E}_\infty$-ring spectra. The results are actually obtained in the generality of equivariant K-theory, KU_G, and the methods apply to obtain the spin orientation, $\mathrm{MSpin} \to \mathrm{KO}$ [26] , as an $\mathbb{E}_\infty$-map as well. The equivariant spin^c orientations constructed here are shown to refine to a map of global orthogonal spectra in [53] .

★ [117] 2006. Kandelaki, *Algebraic K-theory of Fredholm modules and KK-theory.*

Expresses $\mathrm{KK}(A, B)$ as the (nonconnective) algebraic K-theory of an additive category of Fredholm (A, B)-bimodules.

★ [71] 2011. Cortiñas, *Algebraic v. topological K-theory: A friendly match.*
An expository chapter emphasizing comparisons between algebraic and topological K-theory of operator algebras, including the proof of Karoubi's conjecture ([cite Karoubi], [173]).

★ [74] 2015. Dadarlat–Pennig, *Unit spectra of K-theory from strongly self-absorbing C^*-algebras.*
Interprets the group $\mathrm{gl}_1(\mathrm{KU})^1(X)$ as the group of isomorphism classes of locally trivial bundles of C^*-algebras whose fiber is isomorphic to the stable Cuntz algebra, under tensor product. Also provides a similar description of $\mathrm{gl}_1(\mathrm{KU}_{(p)})^1(X)$.

★ [83] 2016. Ebert, *Index theory in spaces of manifolds.*
Generalizes the families index theorem to A-linear Dirac operators valued in the K-theory of a C^*-algebra, A.

★ [155] 2016. Pennig, *A non-commutative model for higher twisted K-theory.*
Models twisted KU-theory for arbitrary twists classified by $\mathrm{BGL}_1(\mathrm{KU})$ in terms of strongly self-absorbing C^*-algebras.

★ [161] 2016. Rosenberg, *Structure and applications of real C^*-algebras.*
An expository survey of real C^*-algebras, including their K-theory. Emphasizes aspects of the theory and applications that are different for the real case. Topics include metrics of positive scalar curvature, orientifold string theory, and the Baum–Connes conjecture.

★ [128] 2018. Land–Nikolaus, *On the relation between K- and L-theory of C^*-algebras.*
Studies comparison maps between topological K-theory and algebraic L-theory of C^*-algebras. Results include:

- There is a bijection
$$\mathrm{ev}_{\mathbb{C}} : \pi_0 \operatorname{Map}_{\mathrm{Fun}(C^*\mathrm{Alg},\mathrm{Sp})}(\mathrm{ku}, \ell) \to \mathbb{Z}\,.$$
There exists $\tau : \mathrm{ku} \to \ell$ that can be made lax symmetric monoidal, and all such τ have $\mathrm{ev}_{\mathbb{C}}(\tau) = 1$.

- There are equivalences of lax symmetric monoidal functors,
$$\mathrm{KU}[\tfrac{1}{2}] \simeq \mathrm{L}[\tfrac{1}{2}] \quad \text{and} \quad \mathrm{KO}[\tfrac{1}{2}] \simeq \mathrm{L}[\tfrac{1}{2}],$$
where in the first equivalence, the functors are on the category of complex C^*-algebras, and in second equivalence, the functors are on the category of real C^*-algebras.

- In the category of spectra, $[\mathrm{KU}, \mathrm{L}(\mathbb{C})] = [\mathrm{L}(\mathbb{C}), \mathrm{KU}] = [\ell(\mathbb{C}), \mathrm{ku}] = 0$.

★ [19] 2019. Gomez Aparicio–Julg–Valette, *The Baum-Connes conjecture: an extended survey.*
An expository survey of the Baum–Connes conjecture.

★ [147] 2019. Mohsen, *Chern–Simons invariants in KK-theory.*
Realizes the Chern–Simons invariants ([68] , [29] , [30] , [31]) as elements of real KK-theory.

★ [169] 2022. Stolz, *Positive scalar curvature – constructions and obstructions.*
An expository survey on the state of the question:

> "Which closed connected smooth manifolds of dimension $n \geq 5$ admit a Riemannian metric whose scalar curvature is everywhere positive?"

as of 2022. The question has been resolved for simply connected manifolds, as well as for various other values of the fundamental group. The particular form of the classification, and the class of fundamental groups for which it has been solved, differs for spin and non-spin manifolds. The following are a combination of many prior results by different people.

- Let M be a non-spin manifold whose universal cover is also non-spin. If the fundamental group of M is one of: $e, (\mathbb{Z}/2)^r, (\mathbb{Z}/p)^r$ for p an odd prime and $r < \dim(M)$, then M admits a positive scalar curvature metric.

- Let M be a spin manifold with fundamental group π, let $f : M \to \mathrm{B}\pi$ be the map classifying the universal cover of M, and let $\alpha_\pi : \mathrm{MSpin}_*(\mathrm{B}\pi) \to \mathrm{KO}_*(\mathrm{B}\pi) \to \mathrm{KO}_*(C^*\pi)$ be the composite of the Atiyah–Bott–Shapiro map [26] on $\mathrm{B}\pi$ and the real assembly map. Then for $\pi = e, \mathbb{Z}/2, \mathbb{Z}/(2k+1), (\mathbb{Z}/p)^r$ for p an odd prime and $r > \dim(M)$, or if π has periodic cohomology, M admits a positive scalar curvature metric if and only if $\alpha_\pi([M, f]) = 0 \in \mathrm{KO}_n(C^*\pi)$.

★ [64] 2023. Bunke, *KK- and E-theory via homotopy theory.*
Constructs stable ∞-categories representing KK-theory and E-theory for C^*-algebras from a homotopy theoretic point of view. These constructions are done by localizing the category of separable C^*-algebras. Develops some of the theory of KK and E using only this categorical construction together with basic properties of C^*-algebras, which allows it to serve as an introduction for homotopy theorists. Reworks the proof of Bott periodicity in [72] in the language of ∞-categories.

★ [129] 2023. Land–Nikolaus–Schlichting, *L-theory of C^*-algebras.*
Studies connections between L-theory of real C^*-algebras and topological K-theory (see [128] for previous results along similar lines). Results include:

- There is a unique symmetric monoidal natural transformation (of functors from real C^*-algebras to spectra)
$$\mathrm{ko} \to \mathrm{L}.$$
- For every real C^*-algebra A, the map
$$\mathrm{ko}(A) \otimes_{\mathrm{ko}} \mathrm{L}(\mathbb{R}) \to \mathrm{L}(A)$$
is an equivalence.

11 Index theory

★ [40] 1964. Atiyah, *K-theory.*
A book that covers the basics of complex K-theory (KU-theory) without assuming knowledge of algebraic topology. The real case (KO) is not discussed, but it is a very useful text for understanding the ideas of both theories. Includes a proof of complex Bott periodicity, and a proof of the Thom isomorphism in KU-theory for complex vector bundles, inducing a complex orientation MU $\to$ KU. See [26] and [23] for the Thom isomorphisms for spin, spin^c, and Real vector bundles, and see [25] for the Thom isomorphisms for Real spin^c vector bundles, as well as for equivariant versions of all of the above Thom isomorphisms. Also contains an appendix proving (independently from [114]) the "Atiyah–Jänich theorem", namely, that the index induces an isomorphism,

$$\text{index} : [X, \mathcal{F}] \to \mathrm{KU}^0(X),$$

where $\mathcal{F}$ is the space of Fredholm operators on a separable complex Hilbert space.

★ [114] 1965. Jänich, *Vektorraumbündel und der Raum der Fredholm Operatoren.*
Proves (independently from [40]) the "Atiyah–Jänich theorem", namely, that the index induces an isomorphism,

$$\text{index} : [X, \mathcal{F}] \to \mathrm{KU}^0(X),$$

where $\mathcal{F}$ is the space of Fredholm operators on a separable complex Hilbert space.

★ [25] 1968. Atiyah, *Bott periodicity and the index of elliptic operators.*
Uses index theory to prove generalizations of the 8-fold Bott periodicity of KO, the 2-fold Bott periodicity of KU, and the $(1,1)$-fold Bott periodicity of KR in the equivariant settings. The most general statement from which all others follow is the following.

- Let $G \to \mathrm{Spin}^c(p,q)$ be a C_2-equivariant homomorphism of groups with C_2-action, and suppose $p \equiv q \mod 8$. Then there is a class $u \in \mathrm{KR}_G(\mathbb{R}^{p,q})$, such that multiplication by u induces an isomorphism,

  $$\mathrm{KR}_G(X) \cong \mathrm{KR}_G(\mathbb{R}^{p,q} \times X).$$

  Here, C_2 acts on $\mathbb{R}^{p,q} = \mathbb{R}^{p+q}$ trivially on the $\mathbb{R}^p$ factor and by -1 on the $\mathbb{R}^q$ factor, and $\mathrm{Spin}^c(p,q)$ is the spin group in the Real Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{R}^{p,q})$ with the induced C_2-action.

The corresponding Thom isomorphisms are formal consequences of the equivariant periodicity theorems.

★ [35] 1968. Atiyah–Singer, *The index of elliptic operators: I.*

Part I of a five part series on the "Atiyah–Singer index theorem". Reviews Thom classes in KU-theory and KU_G-theory via the exterior algebra construction (see [40] and [165]), along with the induced topological index

$$\text{t-ind} : \mathrm{KU}^0_G(TX) \to \mathrm{KU}^0_G(*) \cong \mathrm{RU}(G),$$

where X is a compact smooth G-manifold. Defines the notion of an *index function* to be a natural transformation $\mathrm{KU}^0_{(-)}(T(-)) \to \mathrm{KU}^0_{(-)}(*) \cong \mathrm{RU}(-)$ of functors between appropriately defined categories that satisfies certain axioms. Shows that there is exactly one index function, equal to t-ind. Reviews facts about psuedo-differential operators and elliptic operators over X, and defines the symbol of an elliptic operator as an element of $\mathrm{KU}^0_G(TX)$. Defines the index of a G-invariant[9] elliptic operator P over X as

$$[\mathrm{Ker}(P)] - [\mathrm{Coker}(P)] \in \mathrm{KU}^0_G(*) \cong \mathrm{RU}(G),$$

and explains how the index of P only depends on its symbol in $\mathrm{KU}_G(TX)$. Thus, the index defines a map

$$\text{a-ind} : \mathrm{KU}^0_G(TX) \to \mathrm{KU}^0_G(*) \cong \mathrm{RU}(G),$$

called the analytic index. Shows that a-ind is an index function, thus showing that it equals t-ind,

$$\text{a-ind} = \text{t-ind} : \mathrm{KU}^0_G(TX) \to \mathrm{KU}^0_G(*).$$

As a consequence, a topological formula is obtained for the index of an elliptic operator on a G-manifold in terms of topological K-theory.

★ [37] 1969. Atiyah–Singer, *Index theory for skew-adjoint Fredholm operators.*
Constructs Ω-spectra representing KU and KO out of skew-adjoint Clifford linear Fredholm operators. Includes a proof of Bott periodicity for the spaces in these spectra. The results in this paper can be seen as a generalization of the Atiyah–Jänich theorem (Appendix of [40] , and [114]) from degree 0 K-theory to arbitrary degree.

★ [39] 1971. Atiyah–Singer, *The index of elliptic operators: V.*
Proves a families index theorem for real elliptic operators. This makes essential use of KR-theory.

★ [137] 1971. Matumoto, *Equivariant K-theory and Fredholm operators.*
Generalizes the Atiyah–Jänich theorem (Appendix of [40] , and [114]) to the equivariant setting. Results include:

- Let G be a compact Lie group, and let $\mathcal{F}$ be the G-space of Fredholm operators on the G-Hilbert space $L^2(G, \ell^2)$. Then for any compact G-space X, the G-index map induces an isomorphism,

$$\text{index} : [X, \mathcal{F}]_G \xrightarrow{\sim} \mathrm{KU}^0_G(X).$$

[9] elliptic operators which are not G-invariant can be replaced with a G-invariant one with the same symbol.

For the analogous generalization of [37] in higher degrees, see [FHT?].

★ [80] 1977. Douglas, *Extensions of C^*-algebras and K-homology.*
Part of the book [149] . Describes a homology theory, $\mathrm{Ext}_k(X)$, where $\mathrm{Ext}_1(X)$ is a group of equivalence classes of extensions of the C^*-algebra, $\mathcal{K}$, of compact operators on a separable Hilbert space, by the C^*-algebra, $C(X)$, of continuous complex valued functions on X. Explains that Ext_* is the homology theory dual to KU^* by exhibiting a homomorphism $\mathrm{Ext}_1(X) \to \mathrm{Hom}(\mathrm{KU}^1(X), \mathbb{Z})$, using an index construction. Also explains how $\mathrm{Ext}_1(X)$ carries the structure of a $\mathrm{KU}^0(X)$-module.

★ [149] 1977. Morrel–Singer (Eds.), *K-theory and operator algebras.*
A collection of talks from a conference held at the University of Georgia, April 21 – April 25, 1975, on connections between K-theory and operator algebras. Most of the talks are both expository and exploratory in nature. Contains the references: [80] , [140] , [167] .

★ [158] 1985. Quillen, *Superconnections and the Chern character.*
Introduces the notion of a superconnection on a super ($\mathbb{Z}/2$-graded) vector bundle, and uses it to construct families of differential form representatives for the Chern character. Includes a corresponding geometric description of the Chern character in odd degrees using Clifford linear superconnections. See [135] for a refinement of the Riemann–Roch theorem [27] to an equality of differential forms using this model of the Chern character.

★ [135] 1986. Mathai–Quillen, *Superconnections, Thom classes, and equivariant differential forms.*
Proves a refinement of the differentiable Riemann–Roch theorem [27] to an equality of differential form representatives using the Chern character form constructed in [158] and the Thom classes constructed in [26] .

★ [132] 1989. Lawson–Michelsohn, *Spin geometry.*
A book that introduces the basics of spin geometry, including relations with topological K-theory and applications to topology and geometry. Topics include: a detailed introduction to Clifford algebras and their modules, including the Atiyah–Bott–Shapiro construction, which is extended to the KR case using "Real" Clifford modules; a brief introduction to KU, KO, and KR; spinor bundles and Dirac operators; Thom isomorphisms and the Chern character; various versions of the Atiyah–Singer index theorem; and many applications to topology and differential geometry, including metrics of positive scalar curvature and group actions on manifolds. Also contains useful appendices on principal bundles, classifying spaces, characteristic classes, and Thom classes.

★ [65] 2009. Bunke–Schick, *Smooth K-theory.*
Constructs a multiplicative differential refinement, $\widehat{\mathrm{KU}}$, of KU-theory using geometric cocy-

cles, together with pushforward maps, $p_! : \widehat{\mathrm{KU}}(W) \to \widehat{\mathrm{KU}}(B)$, associated to KU-orientable proper submersions $p : W \to B$, and refines the Chern character to a natural transformation $\widehat{\mathrm{ch}} : \widehat{\mathrm{KU}} \to \widehat{\mathrm{H}\mathbb{Q}}$. Proves a refinement of the Atiyah–Singer index theorem [?] regarding the compatibility of pushforwards with the Chern character.

★ [93] 2010. Freed–Lott, *An index theorem in differential K-theory.*
Refines the Atiyah–Singer index theorem [?] to differential KU-theory. More specifically, for a proper submersion $\pi : X \to B$ of relative dimension n with a Riemannian structure and a differential spin^c-structure, they construct analytic and topological pushforward maps $\mathrm{ind}^{\mathrm{an}}, \mathrm{ind}^{\mathrm{top}} : \widehat{\mathrm{KU}}^0(X) \to \widehat{\mathrm{KU}}^{-n}(B)$, and show that $\mathrm{ind}^{\mathrm{an}} = \mathrm{ind}^{\mathrm{top}}$.

★ [83] 2016. Ebert, *Index theory in spaces of manifolds.*
Generalizes the families index theorem to A-linear Dirac operators valued in the K-theory of a C^*-algebra, A.

★ [178] 2017. Zhang, *A mod 2 index theorem for pin$^-$ manifolds.*
Extends the mod 2 index theorem of Atiyah and Singer [39] to real vector bundles over $(8k+2)$-dimensional compact pin$^-$ manifolds. The analytic index is the reduced η invariant of twisted Dirac operators and the topological index is defined in terms of KO-theory.

★ [19] 2019. Gomez Aparicio–Julg–Valette, *The Baum-Connes conjecture: an extended survey.*
An expository survey of the Baum–Connes conjecture.

★ [147] 2019. Mohsen, *Chern–Simons invariants in KK-theory.*
Realizes the Chern–Simons invariants ([68] , [29] , [30] , [31]) as elements of real KK-theory.

★ [169] 2022. Stolz, *Positive scalar curvature – constructions and obstructions.*
An expository survey on the state of the question:

> "Which closed connected smooth manifolds of dimension $n \geq 5$ admit a Riemannian metric whose scalar curvature is everywhere positive?"

as of 2022. The question has been resolved for simply connected manifolds, as well as for various other values of the fundamental group. The particular form of the classification, and the class of fundamental groups for which it has been solved, differs for spin and non-spin manifolds. The following are a combination of many prior results by different people.

- Let M be a non-spin manifold whose universal cover is also non-spin. If the fundamental group of M is one of: $e, (\mathbb{Z}/2)^r, (\mathbb{Z}/p)^r$ for p an odd prime and $r < \dim(M)$, then M admits a positive scalar curvature metric.

- Let M be a spin manifold with fundamental group π, let $f : M \to \mathrm{B}\pi$ be the map classifying the universal cover of M, and let $\alpha_\pi : \mathrm{MSpin}_*(\mathrm{B}\pi) \to \mathrm{KO}_*(\mathrm{B}\pi) \to \mathrm{KO}_*(C^*\pi)$ be the composite of the Atiyah–Bott–Shapiro map [26] on $\mathrm{B}\pi$ and the real assembly map. Then for $\pi = e, \mathbb{Z}/2, \mathbb{Z}/(2k+1), (\mathbb{Z}/p)^r$ for p an odd prime and $r > \dim(M)$, or if π has periodic cohomology, M admits a positive scalar curvature metric if and only if $\alpha_\pi([M, f]) = 0 \in \mathrm{KO}_n(C^*\pi)$.

★ [87] 2024. Freed, *Index theory on Pin manifolds.*
An exposition of an index theorem for pin manifolds valued in KO-theory. Also includes a refinement to differential KO-theory, modulo the existence of a proof for the differential KO-index theorem (i.e. the KO-theory version of the main result of [93]). The theorem uses specific inclusions of Pin groups into larger Spin groups to apply the corresponding index theory for spin manifolds. Also includes a motivating discussion explaining the relevance of pin manifolds in physics. States that this is based on work with Mike Hopkins.

12 Computations

In this section, we collect references that compute topological K-theory, or something related to topological K-theory, in some context. This can include results of a general flavor that either develop tools for computations or explicitly reduce a class of topological K-theory computations to a different computation. References that *use* topological K-theory to compute something else of interest can be found in Section 13.

★ [56] 1957. Bott, *The stable homotopy of the classical groups.*
Proves Bott periodicity for the first time, using Morse theory.

★ [78] 1959. Dold–Whitney, *Classification of oriented sphere bundles over a 4-complex.*
An analog of [162] in the real case, but it does not discuss the implications for KO-theory explicitly. Work this out.

★ [28] 1961. Atiyah–Hirzebruch, *Vector bundles and homogeneous spaces.*
Develops the basic properties of K-theory as a cohomology theory. Introduces the Atiyah–Hirzebruch spectral sequence. Written in terms of KU, but many constructions carry over to KO. Shows that the Chern character induces an isomorphism,

$$\mathrm{KU}^*(X) \otimes \mathbb{Q} \cong \mathrm{H}^*(X; \mathbb{Q}),$$

where the $*$ represents a $\mathbb{Z}/2$-grading on both sides (on the left side, this is induced by Bott periodicity, and on the right side, it is induced by the $\mathbb{Z}$-grading via the quotient $\mathbb{Z} \to \mathbb{Z}/2$). In particular, $\mathrm{KU}^0(X) \otimes \mathbb{Q} \cong \mathrm{H}^{\mathrm{even}}(X; \mathbb{Q})$ and $\mathrm{KU}^1(X) \otimes \mathbb{Q} \cong \mathrm{H}^{\mathrm{odd}}(X; \mathbb{Q})$. Also proves a special case of the Atiyah–Segal completion theorem [33] , namely when $X = *$ and G is a compact connected Lie group, relating the representation ring of G to $\mathrm{KU}^*(\mathrm{B}G)$.
See [174] for a refinement of the relation between K-theory and ordinary cohomology that includes torsion and an application to KO-theory. See [77] for results on generalizations of the Chern character to arbitrary cohomology theories.

★ [1] 1962. Adams, *Vector fields on spheres.*
Introduces certain power operations in KU-theory and KO-theory (now called "Adams operations") and uses them to determine the maximum number of linearly independent vector fields on each sphere S^n. Notably, the computations involving KO-theory are essential to the proof; consideration of KU-theory alone is insufficient. Also computes stuff about projective spaces. See [24] for a more thorough analysis of power operations in K-theory from the perspective of equivariant K-theory. write some details out here.

★ [77] 1962. Dold, *Relation between ordinary and extraordinary homology.*
Extends the isomorphism, $\mathrm{KU}^*(X) \otimes \mathbb{Q} \cong \mathrm{H}^*(X; \mathbb{Q})$, of [28] to arbitrary cohomology theories, e.g. KO. Results include:

- For any $E \in \mathrm{Sp}$, $E^*(X) \otimes \mathbb{Q} \cong \mathrm{H}^*(X; E_* \otimes \mathbb{Q})$.
- For any $E, F \in \mathrm{Sp}$ such that F_* is a $\mathbb{Q}$-vector space, any map of graded abelian groups $\phi : E_* \to F_*$ extends uniquely to a map $\Phi : E^*(\ ) \to F^*(\ )$ of cohomology theories. If ϕ is multiplicative, then Φ is multiplicative.

For KO, the first result implies $\mathrm{KO}^*(X) \otimes \mathbb{Q} \cong \mathrm{H}^*(X; \mathbb{Q}[\alpha^{\pm 1}])$, where $|\alpha| = 4$. The second result implies that the Chern character is determined by its value at a point.

★ [7] 1964. Anderson, *The real K-theory of classifying spaces.*
Computes $\mathrm{KO}^*(\mathrm{B}G)$ for a compact connected Lie group, G.

$$\mathrm{KO}^i(\mathrm{B}G) \cong \begin{cases} \mathrm{RO}(G)^\wedge, & i = 0 \\ 0, & i = 1 \\ \mathrm{RU}(G)^\wedge / \mathrm{RO}(G)^\wedge, & i = 2 \\ \mathrm{RSp}(G)^\wedge / \mathrm{RU}(G)^\wedge, & i = 3 \\ \mathrm{RSp}(G)^\wedge \cong \mathrm{KSp}^0(\mathrm{B}G), & i = 4 \\ 0, & i = 5 \\ \mathrm{RU}(G)^\wedge / \mathrm{RSp}(G)^\wedge, & i = 6 \\ \mathrm{RO}(G)^\wedge / \mathrm{RU}(G)^\wedge, & i = 7. \end{cases} \quad \bmod 8$$

These computations also follow from [33] .

★ [24] 1966. Atiyah, *Power operations in K-theory.*
Systematically studies power operations in KU-theory via taking (nth) tensor powers of vector bundles viewed as elements in (Σ_n-) equivariant KU-theory, which relates K-theory operations to the representation rings of symmetric groups. Compares the Adams operations of [1] to the Steenrod operations via the Atiyah–Hirzebruch spectral sequence and the Chern character [28] .

★ [70] 1966. Conner–Floyd, *The relation of cobordism to K-theories.*
Establishes the "Conner–Floyd isomorphisms",

$$\mathrm{MU}^*(X) \otimes_{\mathrm{MU}^*} \mathrm{KU}^* \cong \mathrm{KU}^*(X), \quad \mathrm{MSp}^*(X) \otimes_{\mathrm{MSp}^*} \mathrm{KO}^* \cong \mathrm{KO}^*(X);$$

$$\text{or } \mathrm{MU}_*(X) \otimes_{\mathrm{MU}_*} \mathrm{KU}_* \cong \mathrm{KU}_*(X), \quad \mathrm{MSp}_*(X) \otimes_{\mathrm{MSp}_*} \mathrm{KO}_* \cong \mathrm{KO}_*(X).$$

In other words, "complex cobordism determines complex K-theory" and "symplectic cobordism determines KO-theory".

★ [12] 1968. Anderson–Hodgkin, *The K-theory of Eilenberg–MacLane complexes.*

Computes the K-theory of Eilenberg–Mac Lane spaces, $\mathrm{KU}^*(K(\pi,n))$. Also includes an appendix which contains results for $\mathrm{KO}^*(K(\pi,n))$. Write the actual results in both cases.

★ [25] 1968. Atiyah, *Bott periodicity and the index of elliptic operators.*
Uses index theory to prove generalizations of the 8-fold Bott periodicity of KO, the 2-fold Bott periodicity of KU, and the $(1,1)$-fold Bott periodicity of KR in the equivariant settings. The most general statement from which all others follow is the following.

- Let $G \to \mathrm{Spin}^c(p,q)$ be a C_2-equivariant homomorphism of groups with C_2-action, and suppose $p \equiv q \mod 8$. Then there is a class $u \in \mathrm{KR}_G(\mathbb{R}^{p,q})$, such that multiplication by u induces an isomorphism,

$$\mathrm{KR}_G(X) \cong \mathrm{KR}_G(\mathbb{R}^{p,q} \times X).$$

  Here, C_2 acts on $\mathbb{R}^{p,q} = \mathbb{R}^{p+q}$ trivially on the $\mathbb{R}^p$ factor and by -1 on the $\mathbb{R}^q$ factor, and $\mathrm{Spin}^c(p,q)$ is the spin group in the Real Clifford algebra $\mathbb{C}\mathrm{l}(\mathbb{R}^{p,q})$ with the induced C_2-action.

The corresponding Thom isomorphisms are formal consequences of the equivariant periodicity theorems.

★ [165] 1968. Segal, *Equivariant K-theory.*
An introduction to equivariant K-theory built from G-equivariant vector bundles. This work is Segal's thesis, and much of it is attributed to Atiyah.

★ [172] 1968. Stong, *Notes on cobordism theory.*
An exposition of many topics in cobordism theory, including detailed treatments of various relationships between spin and spin^c bordism with topological K-theory (Chapter XI). The following link contains a detailed table of contents which was compiled and made available by Landweber and Ravenel. Contents include:

- a computation of $\mathrm{H}^*(\mathrm{BSpin};\mathbb{Z}/2)$ (p. 291-292) and $\mathrm{H}^*(\mathrm{BSpin}^c;\mathbb{Z}/2)$ (p. 293);
- if a vector bundle admits a KO-orientation, then it must admit a spin structure (p. 301);
- an exposition of the main results of Anderson–Brown–Peterson [9] (p. 302-336);
- all torsion in $\mathrm{H}_*(\mathrm{BSpin};\mathbb{Z})$ and $\mathrm{H}_*(\mathrm{BSpin};\mathbb{Z})$ is of order 2 (p. 316-317);
- two spin manifolds are spin cobordant if and only if they have the same KO-characteristic numbers and the same Stiefel–Whitney numbers (p. 337);
- two spin^c manifolds are spin^c cobordant if and only if they have the same rational characteristic numbers and the same Stiefel–Whitney numbers (p. 337);

- the ring $(\mathrm{MSpin}^c_*/\mathrm{torsion}) \otimes \mathbb{Z}/2$ is a polynomial algebra over $\mathbb{Z}/2$ on generators y_i, with $|y_i| \in \{2, 4k \mid k \in \mathbb{Z}_{\geq 1}\}$ (p. 347);
- after taking homotopy groups, the forgetful map $\mathrm{MU} \to \mathrm{MSpin}^c$ induces a surjection $\mathrm{MU}_* \to \mathrm{MSpin}^c_*/\mathrm{torsion}$ (p. 348);
- if $\eta : \mathbb{S} \to \mathrm{MSpin}$ is the forgetful map (i.e. the unit), then on homotopy groups, $\pi_0\eta$ is an isomorphism, $\pi_n\eta = 0$ for $n \not\equiv 1, 2 \mod 8$, and $\mathrm{Im}(\pi_n\eta) = \mathbb{Z}/2$, for $n \equiv 1, 2 \mod 8$ (p. 350);
- if $\eta^c : \mathbb{S} \to \mathrm{MSpin}^c$ is the forgetful map (i.e. the unit), then on homotopy groups, $\pi_0\eta^c$ is an isomorphism, $\pi_n\eta^c = 0$ for $n > 0$ (p. 350).

★ [33] 1969. Atiyah–Segal, *Equivariant K-theory and completion.*
Proves the "Atiyah–Segal completion theorem" for all compact Lie groups. The theorem concerns the relationship between the equivariant K-theory, $\mathrm{K}^*_G(X)$, of a G-space, X, and the non-equivariant K-theory of its homotopy quotient (a.k.a. Borel construction), $\mathrm{K}^*(X_G) = \mathrm{K}^*(X \times_G \mathrm{E}G)$. In other words, this is the question of whether or not the G-spectrum K_G is "Borel complete", and identifying the corresponding completion map. Results include:

- **Atiyah–Segal completion theorem:** Let G be a compact Lie group and X be a compact G-space. Assume that $\mathrm{KU}^*_G(X)$ is finite over the representation ring $\mathrm{R}(G) = \mathrm{KU}^0_G(*)$. Let I_G be the augmentation ideal in $\mathrm{R}(G)$, and let

$$\mathrm{KU}^*_G(X)^\wedge_{I_G} := \lim_n \Big(\mathrm{KU}^*_G(X)/(I^n_G \cdot \mathrm{KU}^*_G(X)) \Big).$$

  Then the associated bundle construction induces an isomorphism,

$$\mathrm{KU}^*_G(X)^\wedge_{I_G} \cong \mathrm{KU}^*(X_G).$$

- Specializing to the case $X = *$ yields

$$\mathrm{R}(G)^\wedge_{I_G} \cong \mathrm{KU}^0(\mathrm{B}G).$$

- Proves an analogous result for KR-theory, which has implications for KO-theory as well, though they note that the proof does not work if one works entirely with KO-theory. Recovers the computations of [7] .

I don't know whether the finiteness assumption is necessary or not. In [33] , they indicate situations in which it is satisfied, but I often see the condition omitted in other references. In the case $X = *$ and G finite, the completion theorem was proven in [22] , and for $X = *$ and G connected, it was proven in [28] . Note that these statements and proofs did not involve equivariant K-theory.

★ [166] 1971. Segal, *Equivariant stable homotopy theory.*
Introduces $\mathrm{RO}(G)$-graded equivariant stable homotopy theory for a finite group G, as well as an equivariant J-homomorphism, $J : \mathrm{KO}_G^{-1}(X) \to \mathrm{gl}_1 \mathbb{S}_G^0(X)$. Identifies the equivariant stable cohomotopy ring of a point (in degree 0) as the Burside ring, $\mathrm{A}(G)$, of the group G:

$$\mathbb{S}_G^0(*) = \pi_0 \mathbb{S}_G := \operatorname*{colim}_V [S^V, S^V]_G \cong \mathrm{A}(G),$$

where V is a real G-representation, and $\mathrm{A}(G)$ is the Grothendieck group of the category of finite G-sets.

★ [138] 1973. Matumoto, *Equivariant cohomology theories on G-CW copmlexes.*
Develops an Atiyah–Hirzebruch spectral sequence [28] for equivariant cohomology theories, and specifies conditions under which the spectral sequence for KU_G collapses. Applies the spectral sequence to compute KU_G^* of certain types of $\mathrm{O}(n)$-manifolds.

★ [164] 1973. Segal, *The stable homotopy of complex projective space.*
Shows that the inclusion of line bundles into virtual vector bundles,

$$\iota : \mathbb{CP}^\infty = \mathrm{BU}(1) \times \{1\} \to \mathrm{BU} \times \mathbb{Z},$$

induces a split surjection on associated cohomology theories. Specific results:

- Let $P = \Sigma_+^\infty \mathbb{CP}^\infty$. Then for any space X, ι induces a split surjection,

  $$P^0(X) \to \mathrm{KU}^0(X).$$

- There is a homotopy equivalence,

  $$Q(\mathbb{CP}^\infty) = \Omega^\infty P \simeq (\mathrm{BU} \times \mathbb{Z}) \times F,$$

  where the homotopy groups of F are finite.

An analogous result for KO was proven in [47] , and for KR in [150] .

★ [47] 1974. Becker, *Characteristic classes and K-theory.*
Proves an analogue of the results of [164] in the KO case. Results include:

- The inclusion $\mathrm{BO}(2) \to \mathrm{BO}$ induces a split surjection,

  $$(\Sigma_+^\infty \mathrm{BO}(2))^0(X) \to \mathrm{KO}^0(X).$$

See [150] for an analogous result in the KR case.

★ [174] 1974. Thomas, *A relation between K-theory and cohomology.*

Establishes a refinement of the rational isomorphism $\mathrm{KU}^*(X) \otimes \mathbb{Q} \cong \mathrm{H}^{\mathrm{even}}(X; \mathbb{Q})$ of [28] , which includes some torsion, using the theory of λ-rings. Results include:

- For $n \in \mathbb{Z}_{>0}$, let l_n be the set of primes less than n. If X is a CW-complex of dimension at most $4n+3$ and $\mathrm{H}^{4*}(X; \mathbb{Z})$ has no l_{2n+1}-torsion, then there is an isomorphism of abelian groups $\mathrm{KO}(X) \cong \mathrm{H}^{4*}(X; \mathbb{Z})$ modulo 2-torsion.

★ [96] , 1977. Fujii, *On the relation of real cobordism to KR-theory.*
Uses the Real orientation $\mathrm{MU}_{\mathbb{R}} \to \mathrm{KR}$ of [23] to prove a version of the Conner–Floyd isomorphisms [70] for KR-theory. Results include:

- Let (X, A) be a pair of finite C_2-CW complexes. Then $\mathrm{KR}^{0,0}(X, A)$ is a direct summand of $\mathrm{MU}_{\mathbb{R}}^{0,0}(X, A)$.
- There is an isomorphism
$$\mathrm{MU}_{\mathbb{R}}^{\star}(X, A) \otimes_{\mathrm{MU}_{\mathbb{R}}^{\star}} \mathrm{KR}^{\star} \cong \mathrm{KR}^{\star}(X, A),$$
where $\star$ denotes the $\mathrm{RO}(C_2)$-grading.

★ [133] 1977. Madsen–Snaith–Tornehave, *Infinite loop maps in geometric topology.*
Shows that there are equivalences of infinite loop spaces,
$$\mathrm{GL}_1\,\mathrm{KU} \simeq K(\mathbb{Z}/2, 0) \times K(\mathbb{Z}, 2) \times \mathrm{BSU}_{\otimes},$$
and
$$\mathrm{GL}_1\,\mathrm{KO} \simeq K(\mathbb{Z}/2, 0) \times K(\mathbb{Z}/2, 1) \times \mathrm{BSO}_{\otimes}.$$
These results classify (one connected component of) twists of K-theory by the spaces
$$\mathrm{BGL}_1\,\mathrm{KU} \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times \mathrm{BBSU}_{\otimes},$$
and
$$\mathrm{BGL}_1\,\mathrm{KO} \simeq K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2) \times \mathrm{BBSO}_{\otimes}.$$

★ [141] 1979. Meier, *Complex and real K-theory and localization.*
Proves that localization with respect to KU and KO coincide (see also [159] and [58] for further analysis of K-local spectra, i.e. spectra that are local with respect to either KU or KO). More specifically, shows that for any CW complex, X, and any abelian group A, $\widetilde{\mathrm{KU}}_*(X; A) = 0$ if and only if $\widetilde{\mathrm{KO}}_*(X; A) = 0$. Applies this result to conclude that the spaces in the KO-theory Ω-spectrum are KU-local, and vice versa. Similarly, Spin as well as any homogeneous space obtained from the stable classical groups $\mathrm{O}, \mathrm{SO}, \mathrm{U}, \mathrm{SU}, \mathrm{Sp}$, and their loop spaces, are K-local. Computes the homotopy groups of the K-localization of $\mathrm{BU}[2n]$ (the $(2n-1)$-connective cover of BU) for $n \geq 2$.

★ [150] 1982. Nagata–Nishida–Toda, *Segal–Becker theorem for KR-theory.*
Proves an analogue of the results of [164] and [47] for KR-theory. In fact, the results of this paper recover the previous results for KU and KO by forgetting the C_2-action and by taking C_2-fixed points. Results include:

- Let $\mathbb{CP}^\infty$ be the C_2-space given by complex conjugation and let X be a C_2-CW complex. Then there is a split surjection,

$$(\Sigma^\infty_+ \mathbb{CP}^\infty)^0(X) \to \mathrm{KR}^0(X),$$

  where $\Sigma^\infty_+ \mathbb{CP}^\infty$ is the suspension spectrum in C_2-spectra, and where the map is induced by viewing a rank 1 Real line bundle as a Real virtual bundle.

★ [159] 1984. Ravenel, *Localization with respect to certain periodic homology theories.*
Shows that the Bousfield classes of KO and KU are the same, i.e. they induce the same localization functor on the category of spectra (see [141] for an earlier proof, and [58] for an analysis of the K-local category). Proves the "Smash product theorem", i.e. for $p > 2$ and any spectrum X, we have $L_1 X \simeq X \wedge L_1 \mathbb{S}$, where L_1 denotes localization with respect to p-local K-theory.

★ [163] 1987. Rosenberg–Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor.*
Proves a Künneth theorem and a universal coefficient theorem for Kasparov's KK-theory [122] for C^*-algebras. Let $\mathcal{N}$ be the smallest subcategory of the category of separable nuclear C^*-algebras that satisfies the following properties:

1. $\mathcal{N}$ contains all separable C^*-algebras of Type I,
2. $\mathcal{N}$ is closed under strong Morita equivalence (equivalently, stable isomorphism [60]), inductive limits, extensions, and crossed products by $\mathbb{R}$ and $\mathbb{Z}$,
3. if $J \subset A$ is an ideal, then $A, J \in \mathcal{N} \implies A/J \in \mathcal{N}$, and $A, A/J \in \mathcal{N} \implies J \in \mathcal{N}$.

Let $A \in \mathcal{N}$, and let B be a C^*-algebra with a countable approximate unit. Then

- if $\mathrm{K}_*(B)$ is finitely generated, then there is a short exact sequence,

$$0 \to \mathrm{K}^*(A) \otimes \mathrm{K}_*(B) \to \mathrm{KK}_*(A, B) \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{K}^*(A), \mathrm{K}_*(B)) \to 0,$$

- there is a short exact sequence,

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \to \mathrm{KK}_*(A, B) \to \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \to 0,$$

where both short exact sequences are natural in each variable and both split unnaturally.

★ [123] 1988. Kasparov, *Equivariant KK-theory and the Novikov conjecture.*
A follow-up to Kasparov's work on KK-theory [122] . Applies KK-theory to prove the Novikov conjecture in a certain class of cases. more details. The results first appeared in a 1981 preprint which has since been published as [124] in a modified form.

★ [58] 1990. Bousfield. *A classification of K-local spectra.*
Classifies spectra which are K-local, i.e. local with respect to either KU or KO (recall that KU and KO have the same Bousfield class [141] , [159]).

★ [151] 1991. Ochanine, *Elliptic genera, modular forms over* KO_**, and the Brown–Kervaire invariant.*
Shows that, in contrast with [70] ,

$$\mathrm{MSU}^*(X) \otimes_{\mathrm{MSU}^*} \mathrm{KO}^* \not\cong \mathrm{KO}^*(X).$$

need more here. also need more on what in this paper implies the above result.

★ [85] 1995. Fajstrup, *Tate cohomology of periodic K-theory with reality is trivial.*
Shows that the Tate fixed points of KR vanish. This implies that the homotopy and genuine C_2-fixed points of KR are equivalent,

$$\mathrm{KR}^{hC_2} \simeq \mathrm{KR}^{C_2} \simeq \mathrm{KO}\,.$$

★ [124] 1995. Kasparov, *K-theory, group* C^**-algebras, and higher signatures (Conspectus) (first distributed 1981).*
A (modified) preprint version of [123] .

★ [177] 1998. Wolbert, *Classifying modules over K-theory spectra.*
Let K denote either KU or KO. Provides an algebraic criterion for determining when a K-local spectrum is a K-module. Shows that any homotopy K-module is equivalent to K-module, and that any ko_*-module can be realized as the homotopy of a ko-module.

★ [131] 2003. Laures, *An* E_∞ *splitting of spin bordism.*
Establishes a multiplicative splitting,

$$\mathrm{MSpin} \simeq T_\zeta \wedge \bigwedge_{i=1}^{\infty} T\,\mathbb{S},$$

where ζ is the generator of $\pi_{-1}L_{K(1)}\,\mathbb{S}$, T_ζ is the $\mathbb{E}_\infty$-cone over ζ, and $T\,\mathbb{S}$ is the free $\mathbb{E}_\infty$-ring generated by the sphere, $\mathbb{S}$. Derives the following consequences:

- The spin bordism ring, MSpin_* is the free θ-algebra[10] over KO_* in (countably) infinitely many generators, and an algorithm is given for constructing the generators.

- Shows that, $K(1)$-locally, the Atiyah–Bott–Shapiro orientation [26] , $\mathrm{MSpin} \to \mathrm{KO}$, refines to an $\mathbb{E}_\infty$-map (see also [116] and [16]).

- Recovers the Conner–Floyd [70] isomorphism of Hopkins–Hovey [111] .

In the process, also shows that the map

$$\mathrm{KO}^0(\mathrm{MSpin}) \to \mathrm{Hom}_{\mathrm{cts}}(\pi_0\,\mathrm{KO} \wedge \mathrm{MSpin}, \pi_0\,\mathrm{KO} \wedge \mathrm{KO}),$$

is injective, i.e. that a map $\mathrm{MSpin} \to \mathrm{KO}$ is determined by its induced map in KO-homology.

★ [102] 2004. Greenlees, *Equivariant connective K-theory for compact Lie groups.*
Constructs a G-commutative ring spectrum ku_G, for any compact Lie group G, such that

1. the underlying spectrum of ku_G is ku;

2. there is a ring map $\mathrm{MU}_G \to \mathrm{ku}_G$; i.e. ku_G is complex orientable;

3. there is a ring map $\mathrm{ku}_G \to \mathrm{KU}_G$ which is the localization that inverts the Bott element;

4. the coefficient ring ku_G^* is Noetherian, and it's Krull dimension is

$$\max\{2, \text{maximal rank of an abelian subgroup of } G\};$$

5. an analogue of the Atiyah–Segal completion theorem [33] holds, namely,

$$\mathrm{ku}^*(\mathrm{B}G) \cong \mathrm{ku}_G^*(\mathrm{E}G) \cong (\mathrm{ku}_G^*)^\wedge_I,$$

   where $I = \ker(\mathrm{ku}_G^* \to \mathrm{ku}^*)$ is the augmentation ideal.

Also contains partial computations of the coefficient ring ku_G^* for certain groups.

★ [42] 2005. Baker–Richter, *Invertible modules for commutative $\mathbb{S}$-algebras with residue fields.*
Provides general criteria for when the Picard group of a commutative ring spectrum, R, is isomorphic to the Picard group of its homotopy ring, R_*. Examples include $R \in \{\mathrm{KU}, \mathrm{KO}[\frac{1}{2}], \mathrm{ku}, \mathrm{ko}\}$. In particular, this leads to the fact that $\mathrm{Pic}(\mathrm{KU}) \cong \mathbb{Z}/2$. For a computation of $\mathrm{Pic}(\mathrm{KO})$, see [136] .

★ [81] 2005. Dugger, *An Atiyah–Hirzebruch spectral sequence for KR-theory.*

[10] A *θ-algebra* is a commutative algebra over a ring together with a unary operation that satisfies certain properties. The homotopy groups of a $K(1)$-local $\mathbb{E}_\infty$-ring carry the structure of a θ-algebra.

Develops an analogue of the Atiyah–Hirzebruch spectral sequence [28] in the C_2-equivariant setting of KR-theory, using the equivariant Eilenberg–Mac Lane spectra on the constant Mackey functor, $\underline{\mathbb{Z}}$.

★ [16] 2010. Ando–Hopkins–Rezk, *Multiplicative orientations of KO-theory and of the spectrum of topological modular forms.*
Computes π_0 of the space of $\mathbb{E}_\infty$-maps from MSpin to KO (which is also equal to π_0 of the space of $\mathbb{E}_\infty$-maps from MString to KO). Results include:

- There is a bijection between $\pi_0 \, \mathbb{E}_\infty(\mathrm{MSpin}, \mathrm{KO})$ and the set of even sequences $b : \mathbb{Z}_{\geq 4} \to \mathbb{Q}$, such that $b_k = -\dfrac{B_k}{2k} \mod \mathbb{Z}$, where B_k is the kth Bernoulli number, and for each prime p and each $c \in \mathbb{Z}_p^\times$, the sequence $\{(1-c^k)(1-p^{k-1})b_k\}_{k\geq 4}$ satisfies the "generalized Kummer congruences".

- The sequence in the above classification that corresponds to the Atiyah–Bott–Shapiro orientation ([26] , [116]) is $b_k = -\dfrac{B_k}{2k}$.

The proofs are purely homotopical, and uses the machinery developed in [15] .

★ [61] 2010. Bruner–Greenlees, *Connective real K-theory of finite groups.*
A book about equivariant connective real K-theory, ko_G and kr_G, which contains both a significant amount of exposition, as well as many computations. Includes:

- the $\mathrm{RO}(C_2)$-graded homotopy groups of kr;
- an equivalence $(\Sigma^{-\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma\,\mathrm{ko}$;
- an equivalence $(\Sigma^{-2\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma^{-6}\,\mathrm{ko}\langle 4\rangle$;
- an equivalence $(\Sigma^{-3\sigma}\,\mathrm{kr})^{C_2} \simeq \Sigma^{-5}\,\mathrm{ko}\langle 2\rangle$;
- a cofiber sequence $\mathrm{ko} \to \mathrm{kr}^{hC_2} \to \bigvee_{k\geq 1} \Sigma^{-4k}\,\mathrm{H}\,\mathbb{Z}/2$;
- a cofiber sequence $\mathrm{kr}_{hC_2} \to \mathrm{ko} \to \bigvee_{k\geq 0} \Sigma^{4k}\,\mathrm{H}\,\mathbb{Z}/2$;
- the ko homology, $\mathrm{ko}_*(\mathrm{B}G)$, and cohomology, $\mathrm{ko}^*(\mathrm{B}G)$, of the classifying spaces of various groups, including cyclic groups, generalized quaternion groups, the Klein 4-group, dihedral 2-groups, and elementary abelian 2-groups;

★ [62] 2010. Bunke, *Adams operations in smooth K-theory.*

Constructs refinements of the Adams operations [1] for differential (a.k.a. smooth) KU-theory, and proves a corresponding compatibility theorem between the differential Adams operations and pushforward maps for KU-oriented bundles.

★ [18] 2014. Antieau–Gepner–Gómez, *Actions of $K(\pi, n)$ spaces on K-theory and uniqueness of twisted K-theory.*
Proves that

- $[K(\mathbb{Z},3), \mathrm{BGL}_1 \mathrm{KU}] \cong [K(\mathbb{Z},3), K(\mathbb{Z},3)] \cong \mathbb{Z}$, and
- $[K(\mathbb{Z}/2,2), \mathrm{BGL}_1 \mathrm{KO}] \cong [K(\mathbb{Z}/2,2), K(\mathbb{Z}/2,2)] \cong \mathbb{Z}/2$.

This allows one to compare various geometric definitions of twisted KU-theory (resp. KO-theory) twisted by elements of $\mathrm{H}^3(-;\mathbb{Z})$ (resp. $\mathrm{H}^2(-;\mathbb{Z}/2)$) via the homotopy class of the corresponding map $K(\mathbb{Z},3) \to \mathrm{BGL}_1 \mathrm{KU}$ (resp. $K(\mathbb{Z}/2,2) \to \mathrm{BGL}_1 \mathrm{KO}$). In particular, in the real case, all nontrivial ways of defining twisted KO-theory for twists classified by $K(\mathbb{Z}/2,2)$ are equivalent. Also contains an appendix which explains how to adapt the geometric model for KO-theory given in [115] to the twisted case, within the general framework for twisted cohomology set up in [15] , [14] , and [17] .

★ [106] 2014. Heard–Stojanoska, *K-theory, reality, and duality.*
Obtains duality results for KO-theory by interpreting it as both the homotopy fixed points and the homotopy orbits of KR. Also includes a discussion of the differentials in the homotopy fixed point spectral sequence for $\mathrm{KR}^{C_2} \simeq \mathrm{KO}$. Results include:

- There is an equivalence, $\mathrm{KR}_{hC_2} = \mathrm{KU}_{hC_2} \simeq \mathrm{KO}$, between C_2-homotopy orbits of KR, i.e. the homotopy orbits of the complex conjugation action on KU, and KO.
- The Anderson dual of KR is equivalent to $\Sigma^4 \mathrm{KR}$, which implies that the Anderson dual of KO is equivalent to $\Sigma^4 \mathrm{KO} \simeq \mathrm{KSp}$.
- The $K(1)$-local Brown-Comenetz dual of KO is equivalent to $\Sigma^5 \mathrm{KO}$.

★ [160] 2015. Ricka, *Subalgebras of the $\mathbb{Z}/2$-equivariant Steenrod algebra.*
Computes the C_2-equivariant cohomology of kr, i.e. the connective cover of KR, with coefficients in the constant Mackey functor $\underline{\mathbb{Z}/2}$ as

$$\mathrm{H}^\star(\mathrm{kr}; \underline{\mathbb{Z}/2}) \cong \mathcal{A}^\star / \mathcal{E}^\star(2),$$

where $\mathcal{A}^\star$ is the C_2-equivariant mod 2 Steenrod algebra, and $\mathcal{E}^\star(2)$ is a specified subalgebra.

★ [136] 2016. Mathew–Stojanoska, *The Picard group of topological modular forms via descent theory.*

Computes the Picard group $\mathrm{Pic}(\mathrm{KO}) \cong \mathbb{Z}/8$ using C_2-Galois descent for the C_2-Galois extension $\mathrm{KO} \to \mathrm{KU}$, together with the fact that $\mathrm{Pic}(\mathrm{KU}) \cong \mathrm{Pic}(\mathrm{KU}_*) \cong \mathbb{Z}/2$ (see [42]). The computation uses the homotopy fixed point spectral sequence for $\mathfrak{pic}(\mathrm{KU})$ with C_2-action induced from complex conjugation on KU, where $\mathfrak{pic}$ denotes the Picard spectrum.

★ [103] 2020. Guillou–Hill–Isaksen–Ravenel, *The cohomology of C_2-equivariant $\mathcal{A}(1)$ and the homotopy of ko_{C_2}.*
Computes the C_2-equivariant mod 2 cohomology of ko_{C_2}, the C_2-equivariant connective cover of KO_{C_2}, and uses it to compute the $RO(C_2)$-graded homotopy ring of ko_{C_2}. Results include:

- The C_2-fixed point spectrum of ko_{C_2} is $(\mathrm{ko}_{C_2})^{C_2} \simeq \mathrm{ko} \vee \mathrm{ko}$.
- Let σ be the 1-dimensional sign C_2-representation. The C_2-fixed point spectrum of $\Sigma^\sigma \mathrm{ko}_{C_2}$ is $(\Sigma^\sigma \mathrm{ko}_{C_2}) \simeq \mathrm{ko}$.
- The Hopf map induces a C_2-equivariant map $\eta : \Sigma^\sigma \mathrm{ko}_{C_2} \to \mathrm{ko}_{C_2}$, which on C_2-fixed points, induces the map $\eta^{C_2} = (-1, 1) : \mathrm{ko} \to \mathrm{ko} \vee \mathrm{ko}$.
- Complexification $\mathrm{KO}_{C_2} \to \mathrm{KR}$ induces a map on connective covers $c : \mathrm{ko}_{C_2} \to \mathrm{kr}$, which participates in a cofiber sequence,

  $$\Sigma^\sigma \mathrm{ko}_{C_2} \xrightarrow{\eta} \mathrm{ko}_{C_2} \xrightarrow{c} \mathrm{kr}\,.$$

  In terms of spectral Mackey functors [cite, also discuss the spectral Mackey functor form of KR somewhere], this cofiber sequence can be written as

  $$\begin{tikzcd}
  \mathrm{ko} \arrow[r, "(-1,1)"] \arrow[d, bend right, "0"'] & \mathrm{ko} \vee \mathrm{ko} \arrow[r, "\nabla"] \arrow[d, bend right, "\nabla"'] & \mathrm{ko} \arrow[d, bend right, "c"'] \\
  \Sigma^1 \mathrm{ko} \arrow[r, "\eta"'] \arrow[u, bend right, "\eta"'] \arrow[loop below, "\text{sign}"] & \mathrm{ko} \arrow[r, "c"'] \arrow[u, bend right, "\Delta"'] \arrow[loop below, "\text{trivial}"] & \mathrm{ku}, \arrow[u, bend right, "r"'] \arrow[loop below, "\text{conj}"]
  \end{tikzcd}$$

  where $r : \mathrm{ku} \to \mathrm{ko}$ is induced by taking the real vector bundle underlying a complex vector bundle.
- The C_2-equivariant cohomology of ko_{C_2} with coefficients in the constant Mackey functor $\underline{\mathbb{Z}/2}$ is

  $$\mathrm{H}^\star(\mathrm{ko}_{C_2}; \underline{\mathbb{Z}/2}) \cong \mathcal{A}^{C_2} /\!/ \mathcal{A}^{C_2}(1),$$

  where $\mathcal{A}^{C_2}$ is the C_2-equivariant mod 2 Steenrod algebra.
- The geometric fixed points of ko_{C_2} is $(\mathrm{ko}_{C_2})^{gC_2} \simeq \bigvee_{k \geq 0} \Sigma^{4k} \mathrm{H}\hat{\mathbb{Z}}_2$, where $\hat{\mathbb{Z}}_2$ is the 2-adic integers.

★ [45] 2022. Balderrama, *K-theory equivariant with respect to an elementary abelian 2-group.*
Computes the $\mathrm{RO}(G)$-graded homotopy Mackey functor of KU_G and KO_G for $G = (\mathbb{Z}/2)^n$, together with product structures and power operations.

★ [76] 2024. Davis–Wilson, *The cohomology of the connective spectra for K-theory revisited.*
Describes explicit bases for the mod 2 cohomology of connective real and complex K-theory, $\mathrm{H}^*(\mathrm{ko}; \mathbb{Z}/2)$ and $\mathrm{H}^*(\mathrm{ku}; \mathbb{Z}/2)$.

★ [75] 2025. Davis, *The connective KO theory of the Eilenberg-MacLane space $K(\mathbb{Z}/2,2)$.*
Computes $\mathrm{ko}_*(K(\mathbb{Z}/2, 2))$ and $\mathrm{ko}^*(K(\mathbb{Z}/2, 2))$, the connective KO-homology and KO-cohomology of the mod 2 Eilenberg–MacLane space, $K(\mathbb{Z}/2, 2)$, using the Adams spectral sequence. Applies the computation to deduce a new result about Stiefel–Whitney classes of Spin manifolds.

★ [101] 2025. Grady, *On the differential K-theory of moduli stacks.*
Computes the connective differential K-theory and the differential cohomology of the moduli stack of principal G-bundles with connection.

★ [162] 2025. Rosenberg, *Complex K-theory of 4-complexes.*
Deduces special properties of $\mathrm{KU}^0(X)$ when X is a 4-dimensional CW complex. Namely,

- for every complex vector bundle V, the class $[V] \in \mathrm{KU}^0(X)$ is determined by $\mathrm{rank}(V) \in \mathrm{H}^0(X; \mathbb{Z})$, its first Chern class $c_1(V) \in \mathrm{H}^2(X; \mathbb{Z})$, and its second Chern class $c_2(V) \in \mathrm{H}^4(X; \mathbb{Z})$;
- for every element $y \in \mathrm{H}^4(X; \mathbb{Z})$, there exists a unique vector bundle V (up to isomorphism) such that $c_1(V) = 0$, and $c_2(V) = y$;
- the ring $\mathrm{KU}^0(X)$ is determined by the ring $\mathrm{H}^{\mathrm{even}}(X; \mathbb{Z})$; specifically, a generators and relations description of $\mathrm{KU}^0(X)$ is provided, where the generators are taken to be elements of $\mathrm{H}^{\mathrm{even}}(X; \mathbb{Z})$.

The analogous underlying results for the real case can be found in [78] , which does not make explicit reference to topological K-theory.

13 Applications

In this section, we collect references that use topological K-theory to solve mathematical problems that are not internal to topological K-theory. Applications to physics can be found in Section 14.

TODO: Make subsections organizing the applications; e.g. (1) Spin bordism, (2) Homotopy groups of spheres... maybe index theory moves here.

★ [1] 1962. Adams, *Vector fields on spheres.*
Introduces certain power operations in KU-theory and KO-theory (now called "Adams operations") and uses them to determine the maximum number of linearly independent vector fields on each sphere S^n. Notably, the computations involving KO-theory are essential to the proof; consideration of KU-theory alone is insufficient. Also computes stuff about projective spaces. See [24] for a more thorough analysis of power operations in K-theory from the perspective of equivariant K-theory. write some details out here.

★ [5] 1966. Adams, *On the groups J(X)–IV.*
Identifies a family of nontrivial elements in the stable homotopy groups of spheres, $\pi_* \mathbb{S}$, using KO-theory.

★ [10] 1966. Anderson–Brown–Peterson, *Spin cobordism.*
An announcement of [9] . Describes the additive structure and some of the multiplicative structure of the spin cobordism ring, MSpin_*, using KO-theory.

★ [8] 1966. Anderson–Brown–Peterson, *SU-cobordism, KO-characteristic numbers, and the Kervaire invariant.*
Introduces KO-valued characteristic classes and uses them to study the SU-cobordism ring, MSU_*. Specifically, for each $i \in \mathbb{Z}_{\geq 0}$, the i-th KO-Pontrjagin class $\pi^i \in \mathrm{KO}^0(\mathrm{BSO}(2m))$ is defined as follows. Let $\mathrm{U}(1)^m \hookrightarrow \mathrm{SO}(2m)$ be the maximal torus, let X_j be the complex line bundle on $\mathrm{BU}(1)^m$ classified by the projection to the j-th factor of $\mathrm{BU}(1)$, and set $x_j = [X_j] - 1 \in \mathrm{KU}^0(\mathrm{BU}(1)^m)$. Define

$$\pi_t = \prod_{j=1}^{m} (1 + t(x_j + \bar{x}_j)) \in \mathrm{KU}^0(\mathrm{BU}(1)^m)[t],$$

where $\bar{x}_j = \frac{-x_j}{1+x_j}$ denotes the complex conjugate, and let $\tilde{\pi}^i$ be the $\mathrm{KU}^0(\mathrm{BU}(1)^m)$-valued coefficient of t^i in π_t. Then $\tilde{\pi}^i$ lifts to a unique element $\pi^i \in \mathrm{KO}^0(\mathrm{BSO}(2m))$ called the *i-th* KO-*Pontrjagin class.* Results include:

- An explicit comparison between the KO-Pontrjagin classes and the ordinary Pontrjagin classes of a vector bundle via the Pontrjagin (a.k.a. Chern) character map $\mathrm{KO}^0(X) \to \mathrm{H}^*(X; \mathbb{Q})$.

- Identifies when a class $[M] \in \mathrm{MSU}_*$ is zero in terms of the Chern numbers and KO-characteristic numbers of M.
- Identifies which classes in MSU_* contain framed manifolds, in other words, identifies the image of the unit map, $u : \mathbb{S} \to \mathrm{MSU}$, on homotopy groups. More specifically,

$$\mathrm{Im}(u_k) = \begin{cases} \mathbb{Z}/2, & k = 1,2 \\ 0, & k \neq 1,2 \end{cases} \mod 8.$$

- Contains another proof that the Kervaire invariant of an $(8r+2)$-dimensional framed manifold is zero.

The KO-characteristic classes introduced here are also the source of the splitting of MSpin found in [9] .

★ [9] 1967. Anderson–Brown–Peterson, *The structure of the spin cobordism ring.* Contains the results that were announced in [10] . Describes the structure of the spin cobordism ring, MSpin_*, in terms of connective covers of KO-theory and suspensions of $\mathrm{H}\mathbb{Z}/2$. More specifically, let $\pi^i : \mathrm{BSpin} \to \mathrm{KO}$ be the pullback to BSpin of the i-th KO-Pontrjagin class (with the same name) defined in [8] , and let $\pi^I = \pi^{i_1} \dots \pi^{i_k}$, for $I = (i_1, ..., i_k)$. Results include:

- The map $\pi^I : \mathrm{BSpin} \to \mathrm{KO}$ factors as $\pi^I : \mathrm{BSpin} \to \mathrm{ko}\langle n_I \rangle \to \mathrm{KO}$, where

$$n_I = \begin{cases} 4|I|, & |I| \text{ is even} \\ 4|I| - 2, & |I| \text{ is odd,} \end{cases}$$

  where $|I| = i_1 + \cdots + i_k$.

- Let $f_I : \mathrm{MSpin} \to \mathrm{ko}\langle n_I \rangle$ be the map that corresponds to π^I under the Thom isomorphism [26] . There exists a collection of elements $Z \subset \mathrm{H}^*(\mathrm{MSpin}; \mathbb{Z}/2)$, such that the map,

$$F_{\mathrm{ABP}} : \mathrm{MSpin} \xrightarrow{\left(\prod f_I\right) \times \left(\prod z\right)} \prod_{\substack{|I| \text{ even} \\ 1 \notin I}} \mathrm{ko}\langle 4|I| \rangle \times \prod_{\substack{|I| \text{ odd} \\ 1 \notin I}} \mathrm{ko}\langle 4|I| - 2 \rangle \times \prod_{z \in Z} \Sigma^{|z|} \mathrm{H}\,\mathbb{Z}/2,$$

  induces an isomorphism in cohomology with $\mathbb{Z}/2$-coefficients. Since both the domain and the codomain of F_{ABP} are connective with (degree-wise) finitely generated homotopy groups, F_{ABP} is a 2-local equivalence.

- A(n unoriented) manifold M is (unoriented) cobordant to a spin manifold if and only if all Stiefel–Whitney numbers of M involving w_1 or w_2 vanish.

While the complex case is not discussed in this paper, the same methods and constructions apply to produce a 2-local splitting of MSpin^c (see [172] for details). Namely, let π^I also denote the composite $\mathrm{BSpin}^c \to \mathrm{BSO} \xrightarrow{\pi^I} \mathrm{KO} \xrightarrow{\otimes \mathbb{C}} \mathrm{KU}$.

- The map π^I factors as $\pi^I : \mathrm{BSpin}^c \to \mathrm{ku}\langle 4|I|\rangle \to \mathrm{KU}$.
- Let $f_I : \mathrm{MSpin}^c \to \mathrm{ku}\langle 4|I|\rangle$ be the map that corresponds to π^I under the Thom isomorphism [26] . There exists a collection of elements $Z \subset \mathrm{H}^*(\mathrm{MSpin}^c; \mathbb{Z}/2)$, such that the map,

$$F^c_{\mathrm{ABP}} : \mathrm{MSpin}^c \xrightarrow{\left(\prod f_I\right) \times \left(\prod z\right)} \prod_I \mathrm{ku}\langle 4|I|\rangle \times \prod_{z \in Z} \Sigma^{|z|}\, \mathrm{H}\,\mathbb{Z}/2,$$

  is a 2-local equivalence.

★ [130] 1967. Landweber, *Fixed point free conjugations on complex manifolds.*

★ [35] 1968. Atiyah–Singer, *The index of elliptic operators: I.*
Part I of a five part series on the "Atiyah–Singer index theorem". Reviews Thom classes in KU-theory and KU_G-theory via the exterior algebra construction (see [40] and [165]), along with the induced topological index

$$\text{t-ind} : \mathrm{KU}^0_G(TX) \to \mathrm{KU}^0_G(*) \cong \mathrm{RU}(G),$$

where X is a compact smooth G-manifold. Defines the notion of an *index function* to be a natural transformation $\mathrm{KU}^0_{(-)}(T(-)) \to \mathrm{KU}^0_{(-)}(*) \cong \mathrm{RU}(-)$ of functors between appropriately defined categories that satisfies certain axioms. Shows that there is exactly one index function, equal to t-ind. Reviews facts about psuedo-differential operators and elliptic operators over X, and defines the symbol of an elliptic operator as an element of $\mathrm{KU}^0_G(TX)$. Defines the index of a G-invariant[11] elliptic operator P over X as

$$[\mathrm{Ker}(P)] - [\mathrm{Coker}(P)] \in \mathrm{KU}^0_G(*) \cong \mathrm{RU}(G),$$

and explains how the index of P only depends on its symbol in $\mathrm{KU}_G(TX)$. Thus, the index defines a map

$$\text{a-ind} : \mathrm{KU}^0_G(TX) \to \mathrm{KU}^0_G(*) \cong \mathrm{RU}(G),$$

called the analytic index. Shows that a-ind is an index function, thus showing that it equals t-ind,

$$\text{a-ind} = \text{t-ind} : \mathrm{KU}^0_G(TX) \to \mathrm{KU}^0_G(*).$$

As a consequence, a topological formula is obtained for the index of an elliptic operator on a G-manifold in terms of topological K-theory.

[11] elliptic operators which are not G-invariant can be replaced with a G-invariant one with the same symbol.

★ [33] 1969. Atiyah–Segal, *Equivariant K-theory and completion.*
Proves the "Atiyah–Segal completion theorem" for all compact Lie groups. The theorem concerns the relationship between the equivariant K-theory, $\mathrm{K}^*_G(X)$, of a G-space, X, and the non-equivariant K-theory of its homotopy quotient (a.k.a. Borel construction), $\mathrm{K}^*(X_G) = \mathrm{K}^*(X \times_G \mathrm{E}G)$. In other words, this is the question of whether or not the G-spectrum K_G is "Borel complete", and identifying the corresponding completion map. Results include:

- **Atiyah–Segal completion theorem:** Let G be a compact Lie group and X be a compact G-space. Assume that $\mathrm{KU}^*_G(X)$ is finite over the representation ring $\mathrm{R}(G) = \mathrm{KU}^0_G(*)$. Let I_G be the augmentation ideal in $\mathrm{R}(G)$, and let

$$\mathrm{KU}^*_G(X)^\wedge_{I_G} := \lim_n \Big(\mathrm{KU}^*_G(X)/(I^n_G \cdot \mathrm{KU}^*_G(X)) \Big).$$

  Then the associated bundle construction induces an isomorphism,

$$\mathrm{KU}^*_G(X)^\wedge_{I_G} \cong \mathrm{KU}^*(X_G).$$

- Specializing to the case $X = *$ yields

$$\mathrm{R}(G)^\wedge_{I_G} \cong \mathrm{KU}^0(\mathrm{B}G).$$

- Proves an analogous result for KR-theory, which has implications for KO-theory as well, though they note that the proof does not work if one works entirely with KO-theory. Recovers the computations of [7] .

I don't know whether the finiteness assumption is necessary or not. In [33] , they indicate situations in which it is satisfied, but I often see the condition omitted in other references. In the case $X = *$ and G finite, the completion theorem was proven in [22] , and for $X = *$ and G connected, it was proven in [28] . Note that these statements and proofs did not involve equivariant K-theory.

★ [164] 1973. Segal, *The stable homotopy of complex projective space.*
Shows that the inclusion of line bundles into virtual vector bundles,

$$\iota : \mathbb{CP}^\infty = \mathrm{BU}(1) \times \{1\} \to \mathrm{BU} \times \mathbb{Z},$$

induces a split surjection on associated cohomology theories. Specific results:

- Let $P = \Sigma^\infty_+ \mathbb{CP}^\infty$. Then for any space X, ι induces a split surjection,

$$P^0(X) \to \mathrm{KU}^0(X).$$

- There is a homotopy equivalence,

$$Q(\mathbb{CP}^\infty) = \Omega^\infty P \simeq (\mathrm{BU} \times \mathbb{Z}) \times F,$$

  where the homotopy groups of F are finite.

An analogous result for KO was proven in [47] , and for KR in [150] .

★ [47] 1974. Becker, *Characteristic classes and K-theory.*
Proves an analogue of the results of [164] in the KO case. Results include:

- The inclusion $\mathrm{BO}(2) \to \mathrm{BO}$ induces a split surjection,

$$(\Sigma^\infty_+ \mathrm{BO}(2))^0(X) \to \mathrm{KO}^0(X).$$

See [150] for an analogous result in the KR case.

★ [141] 1979. Meier, *Complex and real K-theory and localization.*
Proves that localization with respect to KU and KO coincide (see also [159] and [58] for further analysis of K-local spectra, i.e. spectra that are local with respect to either KU or KO). More specifically, shows that for any CW complex, X, and any abelian group A, $\widetilde{\mathrm{KU}}_*(X;A) = 0$ if and only if $\widetilde{\mathrm{KO}}_*(X;A) = 0$. Applies this result to conclude that the spaces in the KO-theory Ω-spectrum are KU-local, and vice versa. Similarly, Spin as well as any homogeneous space obtained from the stable classical groups $\mathrm{O}, \mathrm{SO}, \mathrm{U}, \mathrm{SU}, \mathrm{Sp}$, and their loop spaces, are K-local. Computes the homotopy groups of the K-localization of $\mathrm{BU}[2n]$ (the $(2n-1)$-connective cover of BU) for $n \geq 2$.

★ [150] 1982. Nagata–Nishida–Toda, *Segal–Becker theorem for KR-theory.*
Proves an analogue of the results of [164] and [47] for KR-theory. In fact, the results of this paper recover the previous results for KU and KO by forgetting the C_2-action and by taking C_2-fixed points. Results include:

- Let $\mathbb{CP}^\infty$ be the C_2-space given by complex conjugation and let X be a C_2-CW complex. Then there is a split surjection,

$$(\Sigma^\infty_+ \mathbb{CP}^\infty)^0(X) \to \mathrm{KR}^0(X),$$

  where $\Sigma^\infty_+ \mathbb{CP}^\infty$ is the suspension spectrum in C_2-spectra, and where the map is induced by viewing a rank 1 Real line bundle as a Real virtual bundle.

★ [98] 1984. Giambalvo–Pengelley, *The homology of MSpin.*
Describes the mod 2 homology of MSpin as a comodule algebra over the dual Steenrod algebra. Includes applications to the structure of the spin bordism ring, MSpin_*.

★ [135] 1986. Mathai–Quillen, *Superconnections, Thom classes, and equivariant differential forms.*
Proves a refinement of the differentiable Riemann–Roch theorem [27] to an equality of differential form representatives using the Chern character form constructed in [158] and the Thom classes constructed in [26] .

★ [41] 1987. Bahri–Gilkey, *Pin^c cobordism and equivariant Spin^c cobordism of cyclic 2-groups.*
Computes the Pin^c bordism groups, MPin^c_*, as well as $\mathrm{MSpin}^c_*(\mathrm{B}(\mathbb{Z}/2^\nu))$.

★ [123] 1988. Kasparov, *Equivariant KK-theory and the Novikov conjecture.*
A follow-up to Kasparov's work on KK-theory [122] . Applies KK-theory to prove the Novikov conjecture in a certain class of cases. more details. The results first appeared in a 1981 preprint which has since been published as [124] in a modified form.

★ [132] 1989. Lawson–Michelsohn, *Spin geometry.*
A book that introduces the basics of spin geometry, including relations with topological K-theory and applications to topology and geometry. Topics include: a detailed introduction to Clifford algebras and their modules, including the Atiyah–Bott–Shapiro construction, which is extended to the KR case using "Real" Clifford modules; a brief introduction to KU, KO, and KR; spinor bundles and Dirac operators; Thom isomorphisms and the Chern character; various versions of the Atiyah–Singer index theorem; and many applications to topology and differential geometry, including metrics of positive scalar curvature and group actions on manifolds. Also contains useful appendices on principal bundles, classifying spaces, characteristic classes, and Thom classes.

★ [111] 1992. Hopkins–Hovey, *Spin cobordism determines real K-theory.*
Proves analogues of the "Conner–Floyd isomorphisms" [70] for spin and spin^c bordism, i.e.

$$(\mathrm{MSpin}^c)^*(X) \otimes_{(\mathrm{MSpin}^c)^*} \mathrm{KU}^* \cong \mathrm{KU}^*(X), \quad \mathrm{MSpin}^*(X) \otimes_{\mathrm{MSpin}^*} \mathrm{KO}^* \cong \mathrm{KO}^*(X);$$

$$\text{or } \mathrm{MSpin}^c_*(X) \otimes_{\mathrm{MSpin}^c_*} \mathrm{KU}_* \cong \mathrm{KU}_*(X), \quad \mathrm{MSpin}_*(X) \otimes_{\mathrm{MSpin}_*} \mathrm{KO}_* \cong \mathrm{KO}_*(X).$$

Note that the proof in this paper does not use a geometric model for K-theory whose (co)cycles are spin(^{c}) manifolds to establish these isomorphisms. Rather, the proof proceeds algebraically by localizing at each prime.

★ [124] 1995. Kasparov, *K-theory, group C^*-algebras, and higher signatures (Conspectus) (first distributed 1981).*
A (modified) preprint version of [123] .

★ [131] 2003. Laures, *An E_∞ splitting of spin bordism.*
Establishes a multiplicative splitting,

$$\mathrm{MSpin} \simeq T_\zeta \wedge \bigwedge_{i=1}^{\infty} T\,\mathbb{S},$$

where ζ is the generator of $\pi_{-1}L_{K(1)}\,\mathbb{S}$, T_ζ is the $\mathbb{E}_\infty$-cone over ζ, and $T\,\mathbb{S}$ is the free $\mathbb{E}_\infty$-ring generated by the sphere, $\mathbb{S}$. Derives the following consequences:

- The spin bordism ring, MSpin_* is the free θ-algebra[12] over KO_* in (countably) infinitely many generators, and an algorithm is given for constructing the generators.
- Shows that, $K(1)$-locally, the Atiyah–Bott–Shapiro orientation [26] , $\mathrm{MSpin} \to \mathrm{KO}$, refines to an $\mathbb{E}_\infty$-map (see also [116] and [16]).
- Recovers the Conner–Floyd [70] isomorphism of Hopkins–Hovey [111] .

In the process, also shows that the map

$$\mathrm{KO}^0(\mathrm{MSpin}) \to \mathrm{Hom}_{\mathrm{cts}}(\pi_0\,\mathrm{KO} \wedge \mathrm{MSpin}, \pi_0\,\mathrm{KO} \wedge \mathrm{KO}),$$

is injective, i.e. that a map $\mathrm{MSpin} \to \mathrm{KO}$ is determined by its induced map in KO-homology.

★ [161] 2016. Rosenberg, *Structure and applications of real C^*-algebras.*
An expository survey of real C^*-algebras, including their K-theory. Emphasizes aspects of the theory and applications that are different for the real case. Topics include metrics of positive scalar curvature, orientifold string theory, and the Baum–Connes conjecture.

★ [128] 2018. Land–Nikolaus, *On the relation between K- and L-theory of C^*-algebras.*
Studies comparison maps between topological K-theory and algebraic L-theory of C^*-algebras. Results include:

- There is a bijection
$$\mathrm{ev}_{\mathbb{C}} : \pi_0\,\mathrm{Map}_{\mathrm{Fun}(C^*\mathrm{Alg},\mathrm{Sp})}(\mathrm{ku}, \ell) \to \mathbb{Z}\,.$$
There exists $\tau : \mathrm{ku} \to \ell$ that can be made lax symmetric monoidal, and all such τ have $\mathrm{ev}_{\mathbb{C}}(\tau) = 1$.
- There are equivalences of lax symmetric monoidal functors,
$$\mathrm{KU}[\tfrac{1}{2}] \simeq \mathrm{L}[\tfrac{1}{2}] \quad \text{and} \quad \mathrm{KO}[\tfrac{1}{2}] \simeq \mathrm{L}[\tfrac{1}{2}],$$
where in the first equivalence, the functors are on the category of complex C^*-algebras, and in second equivalence, the functors are on the category of real C^*-algebras.
- In the category of spectra, $[\mathrm{KU}, \mathrm{L}(\mathbb{C})] = [\mathrm{L}(\mathbb{C}), \mathrm{KU}] = [\ell(\mathbb{C}), \mathrm{ku}] = 0$.

★ [169] 2022. Stolz, *Positive scalar curvature – constructions and obstructions.*
An expository survey on the state of the question:

> “Which closed connected smooth manifolds of dimension $n \geq 5$ admit a Riemannian metric whose scalar curvature is everywhere positive?”

[12] A *θ-algebra* is a commutative algebra over a ring together with a unary operation that satisfies certain properties. The homotopy groups of a $K(1)$-local $\mathbb{E}_\infty$-ring carry the structure of a θ-algebra.

as of 2022. The question has been resolved for simply connected manifolds, as well as for various other values of the fundamental group. The particular form of the classification, and the class of fundamental groups for which it has been solved, differs for spin and non-spin manifolds. The following are a combination of many prior results by different people.

- Let M be a non-spin manifold whose universal cover is also non-spin. If the fundamental group of M is one of: $e, (\mathbb{Z}/2)^r, (\mathbb{Z}/p)^r$ for p an odd prime and $r < \dim(M)$, then M admits a positive scalar curvature metric.

- Let M be a spin manifold with fundamental group π, let $f : M \to \mathrm{B}\pi$ be the map classifying the universal cover of M, and let $\alpha_\pi : \mathrm{MSpin}_*(\mathrm{B}\pi) \to \mathrm{KO}_*(\mathrm{B}\pi) \to \mathrm{KO}_*(C^*\pi)$ be the composite of the Atiyah–Bott–Shapiro map [26] on $\mathrm{B}\pi$ and the real assembly map. Then for $\pi = e, \mathbb{Z}/2, \mathbb{Z}/(2k+1), (\mathbb{Z}/p)^r$ for p an odd prime and $r > \dim(M)$, or if π has periodic cohomology, M admits a positive scalar curvature metric if and only if $\alpha_\pi([M, f]) = 0 \in \mathrm{KO}_n(C^*\pi)$.

★ [129] 2023. Land–Nikolaus–Schlichting, *L-theory of C^*-algebras.*
Studies connections between L-theory of real C^*-algebras and topological K-theory (see [128] for previous results along similar lines). Results include:

- There is a unique symmetric monoidal natural transformation (of functors from real C^*-algebras to spectra)
$$\mathrm{ko} \to \mathrm{L}.$$

- For every real C^*-algebra A, the map
$$\mathrm{ko}(A) \otimes_{\mathrm{ko}} \mathrm{L}(\mathbb{R}) \to \mathrm{L}(A)$$
is an equivalence.

★ [75] 2025. Davis, *The connective KO theory of the Eilenberg-MacLane space K($\mathbb{Z}/2$,2).*
Computes $\mathrm{ko}_*(K(\mathbb{Z}/2, 2))$ and $\mathrm{ko}^*(K(\mathbb{Z}/2, 2))$, the connective KO-homology and KO-cohomology of the mod 2 Eilenberg–MacLane space, $K(\mathbb{Z}/2, 2)$, using the Adams spectral sequence. Applies the computation to deduce a new result about Stiefel–Whitney classes of Spin manifolds.

14 Connections with physics

★ [86] 2000. Freed, *Dirac charge quantization and generalized differential cohomology.*
Discusses physical motivations for generalized differential cohomology. Argues that currents associated to quantized charges for an abelian gauge field should be classified by a generalized cohomology theory equipped with a map to real valued cohomology, $c : E \to \mathrm{H}\mathbb{R}$, together with the additional data of differential forms that are compatible with the map c via de Rham cohomology, in an appropriate sense. See [112] for the underlying mathematical theory of generalized differential cohomology that was being developed at around the same time.

★ [88] 2000. Freed–Hopkins, *On Ramond-Ramond fields and K-theory.*
Builds off of [148] 's observation that Ramond-Ramond fields in type-II string theory have topological invariants in K-theory by suggesting that Ramond-Ramond fields are classified by *differential* K-theory, thus leading to the development of differential K-theory in the first place.

★ [148] 2000. Moore–Witten, *Self-duality, Ramond-Ramond fields, and K-theory.*
Shows that Ramond-Ramond fields in type-I and type-II string theory determine classes in K-theory. More specifically, Ramond-Ramond fields in type-I string theory on X induce elements of $\mathrm{KO}^{-1}(X)$, Ramond-Ramond fields in type-IIA string theory on X induce elements of $\mathrm{KU}^0(X)$, and Ramond-Ramond fields in type-IIB string theory on X induce elements of $\mathrm{KU}^1(X)$. For a refinement of this idea to differential K-theory, see [88] .

★ [112] 2005. Hopkins–Singer, *Quadratic functions in geometry, topology, and M-theory.*
Constructs differential refinements for arbitrary cohomology theories, and includes a brief discussion about differential K-theory. Includes a discussion of Anderson duality (also see [106]). Add details.

★ [95] 2007. Freed–Moore–Segal, *The uncertainty of fluxes.*
Studies the Anderson duality of KO, as in [112] and [106] .

★ [125] 2009. Kitaev, *Periodic table for topological insulators and superconductors.*
Classifies gapped phases of noninteracting fermions in terms of KU-theory and KO-theory using a version of the Atiyah–Bott–Shapiro construction [26] relating Clifford algebras to K-theory as developed in the book [119] .

★ [110] 2010. Hohnhold–Stolz–Teichner, *From minimal geodesics to super symmetric field theories.*
Describes equivalences between eight models for the spaces in (an Ω-spectrum model for) the KO-theory spectrum, including new ones related to supersymmetric Euclidean field theories.

These spaces can be roughly describes as follows.

1. Bott's spaces of minimal geodesics [56] .
2. Milnor's spaces of Clifford module structures [145] .
3. Spaces of Clifford-linear unbounded operators.
4. Configuration spaces of finitely many points in a circle labeled by finite dimensional Clifford modules.
5. Spaces of super semigroups of finite rank Clifford-linear operators.
6. Spaces of supersymmetric positive Euclidean field theories of dimension $1|1$.
7. The classifying spaces of certain Quillen-style categories that "categorify" the Atiyah–Bott–Shapiro description of KO_* [26] .
8. Atiyah–Singer's spaces of Clifford-linear Fredholm operators [37] .

Also describes a symmetric ring spectrum representing KO-theory similar to the one in [115] , but using unbounded operators (3) instead of Fredholm operators (8).

★ [161] 2016. Rosenberg, *Structure and applications of real C^*-algebras.*
An expository survey of real C^*-algebras, including their K-theory. Emphasizes aspects of the theory and applications that are different for the real case. Topics include metrics of positive scalar curvature, orientifold string theory, and the Baum–Connes conjecture.

★ [100] 2023. Gomi–Yamashita, *Differential KO-theory via gradations and mass terms.*
Refines the model of KO-theory developed in [119] in terms of gradations on Clifford modules to a model of differential KO-theory interpreted as "fermionic mass terms". Also develops this model for twisted differential KO-theory.

★ [87] 2024. Freed, *Index theory on Pin manifolds.*
An exposition of an index theorem for pin manifolds valued in KO-theory. Also includes a refinement to differential KO-theory, modulo the existence of a proof for the differential KO-index theorem (i.e. the KO-theory version of the main result of [93]). The theorem uses specific inclusions of Pin groups into larger Spin groups to apply the corresponding index theory for spin manifolds. Also includes a motivating discussion explaining the relevance of pin manifolds in physics. States that this is based on work with Mike Hopkins.

15 Under construction

★ [78] 1959. Dold–Whitney, *Classification of oriented sphere bundles over a 4-complex.*
An analog of [162] in the real case, but it does not discuss the implications for KO-theory explicitly. Work this out.

★ [2] 1963. Adams, *On the groups J(X)–I.*

★ [34] 1963. Atiyah–Singer, *The index of elliptic operators on compact manifolds.*
An announcement of the “Atiyah–Singer index theorem”. Specify which results from the 5 papers are in here.

★ [3] 1965. Adams, *On the groups J(X)–II.*

★ [4] 1965. Adams, *On the groups J(X)–III.*

★ [5] 1966. Adams, *On the groups J(X)–IV.*
Identifies a family of nontrivial elements in the stable homotopy groups of spheres, $\pi_* \mathbb{S}$, using KO-theory.

★ [130] 1967. Landweber, *Fixed point free conjugations on complex manifolds.*

★ [32] 1968. Atiyah–Segal, *The index of elliptic operators: II.*

★ [36] 1968. Atiyah–Singer, *The index of elliptic operators: III.*

★ [11] 1969. Anderson, *Universal coefficient theorems for K-theory.*
Introduces “Anderson duality”. Shows that KU is its own Anderson dual, and that KO is the Anderson dual of KSp $\simeq \Sigma^4$ KO (see [106] for a more recent treatment using KR-theory). What in this note shows this? Also figure out what other concrete results are obtained here in a way that can be written efficiently.

★ [38] 1971. Atiyah–Singer, *The index of elliptic operators: IV.*
Proves the index theorem for families of complex elliptic operators.

★ [39] 1971. Atiyah–Singer, *The index of elliptic operators: V.*

Proves a families index theorem for real elliptic operators. This makes essential use of KR-theory.

★ [68] 1974. Chern–Simons, *Characteristic forms and geometric invariants.*
Introduces the "Chern–Simons" invariants, which is refined to K-theory in [29] , [30] , and [31] .

★ [29] 1975. Atiyah–Patodi–Singer, *Spectral asymmetry and Riemannian geometry. I.*

★ [30] 1975. Atiyah–Patodi–Singer, *Spectral asymmetry and Riemannian geometry. II.*

★ [31] 1976. Atiyah–Patodi–Singer, *Spectral asymmetry and Riemannian geometry. III.*

★ [167] 1977. Segal, *K-homology theory and algebraic K-theory.*
Part of the book [149] .

★ [135] 1986. Mathai–Quillen, *Superconnections, Thom classes, and equivariant differential forms.*
Proves a refinement of the differentiable Riemann–Roch theorem [27] to an equality of differential form representatives using the Chern character form constructed in [158] and the Thom classes constructed in [26] .

★ [109] 1990. Higson, *Categories of fractions and excision in KK-theory.*
Introduces E-theory of C^*-algebras as a close approximation of KK-theory that satisfies excision. Provides a universal property for E-theory that is similar to the one for KK-theory given in [107] .

★ [97] 1994. Getzler, *The equivariant Chern character for non-compact Lie groups.*
Constructs a model for the equivariant Chern character of an equivariant vector bundle along the lines of the Chern–Weil construction in the non-equivariant case.

★ [86] 2000. Freed, *Dirac charge quantization and generalized differential cohomology.*
Discusses physical motivations for generalized differential cohomology. Argues that currents associated to quantized charges for an abelian gauge field should be classified by a generalized cohomology theory equipped with a map to real valued cohomology, $c : E \to \mathrm{H}\mathbb{R}$, together with the additional data of differential forms that are compatible with the map c via de Rham cohomology, in an appropriate sense. See [112] for the underlying mathematical theory of generalized differential cohomology that was being developed at around the same

time.

★ [52] 2002. Boersema, *Real C*-algebras, united K-theory, and the Künneth formula.*

★ [170] 2004. Stolz–Teichner, *What is an elliptic object?.*

★ [69] 2006. Cheung, *Supersymmetric field theories and generalized cohomology.*

★ [143] 2006. Meyer–Nest, *The Baum–Connes conjecture via localisation of categories.*

★ [95] 2007. Freed–Moore–Segal, *The uncertainty of fluxes.*
Studies the Anderson duality of KO, as in [112] and [106] .

★ [59] 2008. Brodzki–Mathai–Rosenberg–Szabo, *D-branes, KK-theory and duality on noncommutative spaces.*

★ [89] 2008. Freed–Hopkins–Teleman, *Twisted equivariant K-theory with complex coefficients.*

★ [168] 2008. Simons–Sullivan, *Structured vector bundles define differential K-theory.*

★ [90] 2011. Freed–Hopkins–Teleman, *Loop groups and twisted K-theory I.*
The first part of a three paper series, along with [92] and [91] .

★ [91] 2011. Freed–Hopkins–Teleman, *Loop groups and twisted K-theory III.*
The third part of a three paper series, along with [90] and [92] .

★ [171] 2011. Stolz–Teichner, *Supersymmetric field theories and generalized cohomology.*

★ [92] 2013. Freed–Hopkins–Teleman, *Loop groups and twisted K-theory II.*
The second part of a three paper series, along with [90] and [91] .

★ [94] 2013. Freed–Moore, *Twisted equivariant matter.*
Relates topological phases of quantum systems to a version of twisted equivariant K-theory.

★ [83] 2016. Ebert, *Index theory in spaces of manifolds.*
Generalizes the families index theorem to A-linear Dirac operators valued in the K-theory of a C^*-algebra, A.

★ [155] 2016. Pennig, *A non-commutative model for higher twisted K-theory.*
Models twisted KU-theory for arbitrary twists classified by $\mathrm{BGL}_1(\mathrm{KU})$ in terms of strongly self-absorbing C^*-algebras.

★ [156] 2018. Peterson, *Formal geometry and bordism operations.*
A book emphasizing an algebro-geometric approach to chromatic homotopy theory. Contains an exposition of [16] 's homotopical construction of an $\mathbb{E}_\infty$-map MSpin $\to$ KO.

★ [147] 2019. Mohsen, *Chern–Simons invariants in KK-theory.*
Realizes the Chern–Simons invariants ([68] , [29] , [30] , [31]) as elements of real KK-theory.

★ [99] 2021. Gomi, *Freed-Moore K-theory.*
Contains a detailed treatment of the version of K-theory used and developed in [94] , both in terms of Fredholm operators [37] and in terms of Karoubi's model [119] .

★ [46] 2022. Barthel–Berwick-Evans–Stapleton, *Power operations in the Stolz–Teichner program.*

★ [54] 2022. Bohmann–Hazel–Ishak–Kędziorek–May, *Genuine-commutative structure on rational equivariant K-theory for finite abelian groups.*

★ [55] 2022. Bohmann–Hazel–Ishak–Kędziorek–May, *Naive-commutative ring structure on rational equivariant K-theory for abelian groups.*

★ [121] 2023. Karoubi, *Real K-theories.*

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